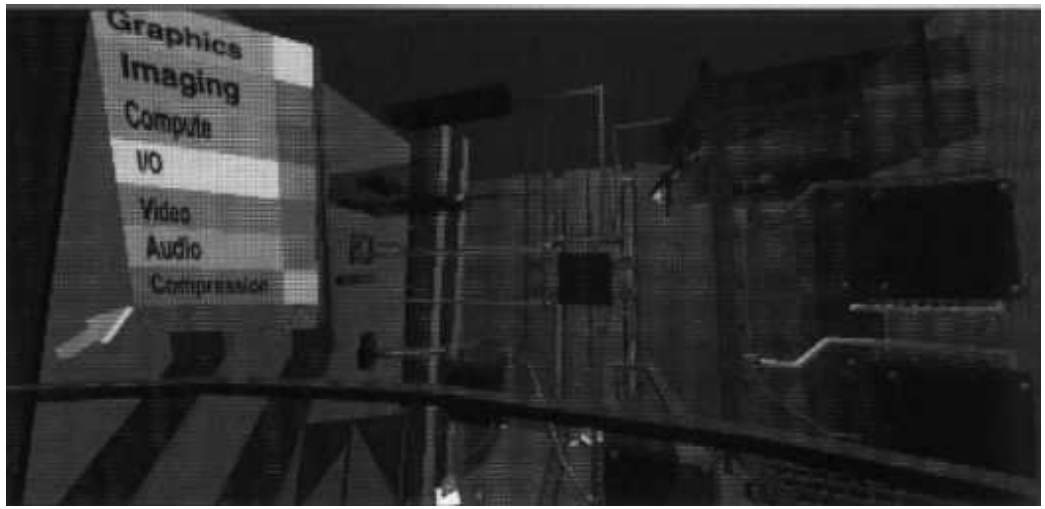
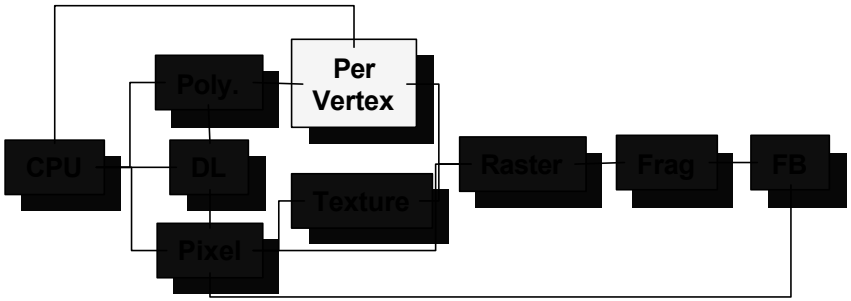
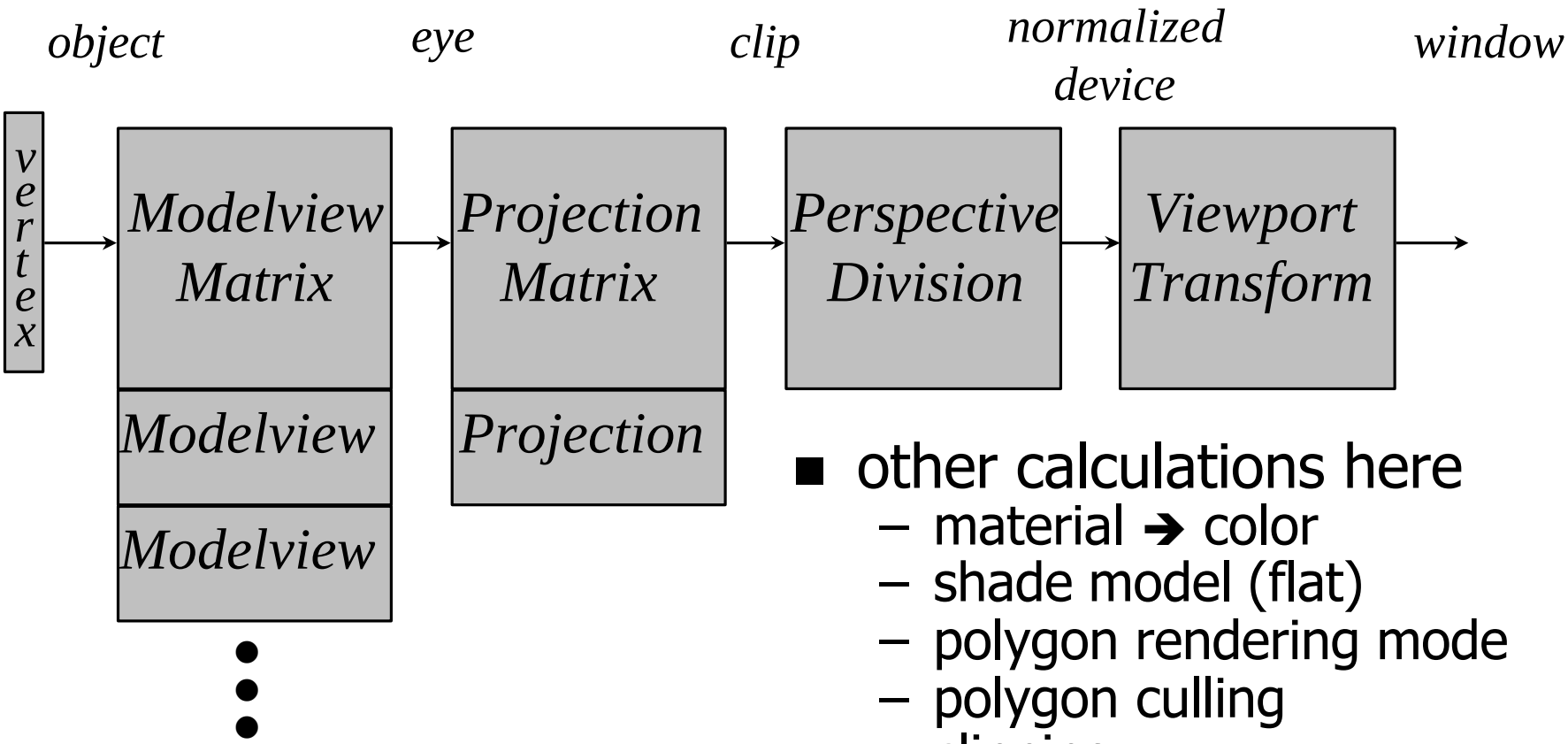
Realtime 3D Computer Graphics & Virtual Reality



Viewing

Transformation Pipeline





- other calculations here
  - material ➔ color
  - shade model (flat)
  - polygon rendering mode
  - polygon culling
  - clipping

Example Frame Rendering

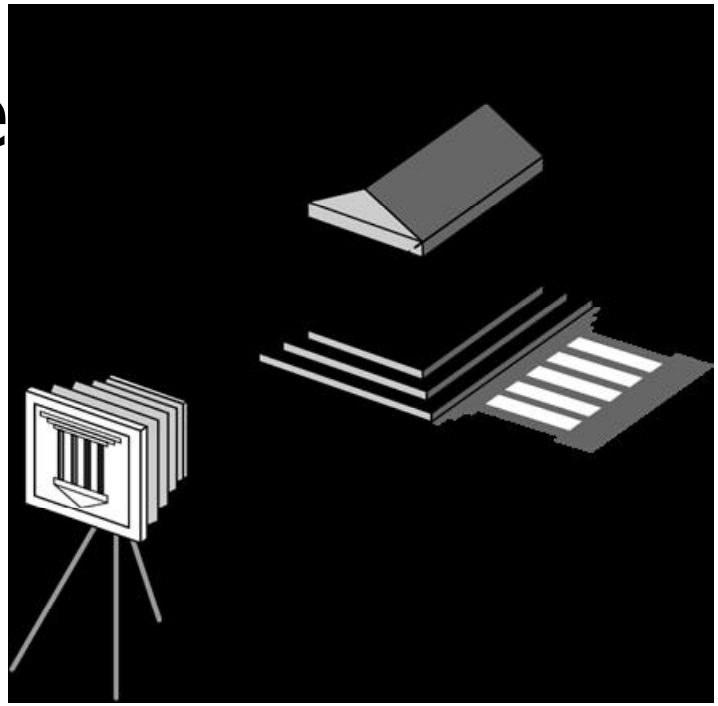
- Establish World
- **Establish Viewpoint & Display Plane**
- **Perform 3D Clipping**
- **Projection of World onto Display Plane**
- Perform Hidden Surface Removal
- Determine Display Colors
- Rasterization of 2D Display

*Rendering order may be changed based on algorithms involved!

Classical and General Viewing

Viewing

- Process for “Seeing” a world
- Projection of a 3D world onto a 2D plane
- Synthetic Camera Mode



Viewing Issues

- Location of viewer
- Location of view plane
- What can be seen (Culling and Clipping)
- How relationships are maintained
  - Parallel Lines
  - Angles
  - Distances (Foreshortening)
  - Relation to Viewer

Objects vs. Scenes

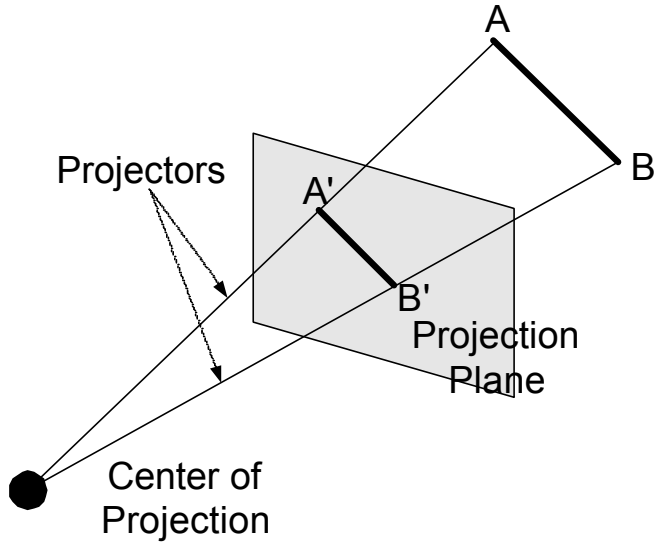
- Some viewing techniques better suited for viewing single objects rather than entire scenes
- Viewing an object from the outside (external viewing)
  - Engineering, External Buildings
- Viewing an object from within (internal viewing)
  - Internal Buildings, Games

Definitions

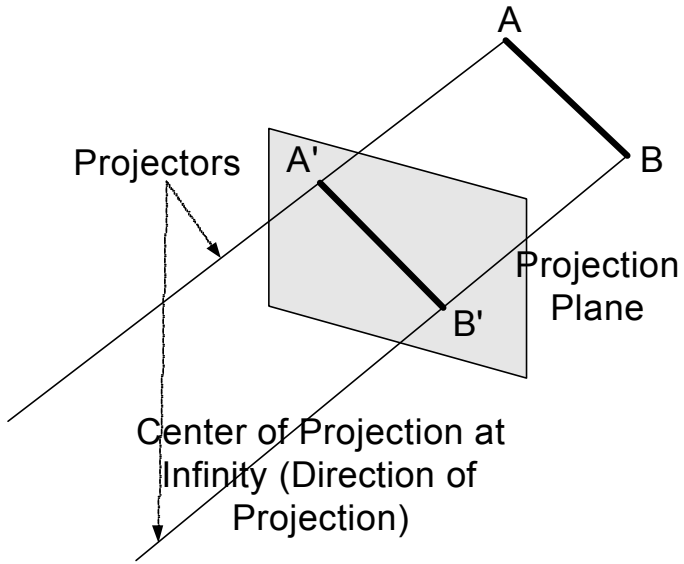
- Projection: a transformation that maps from a higher dimensional space to a lower dimensional space (e.g. 3D->2D)
- Center of projection (CoP): the position of the eye or camera with respect to which the projection is performed (also eye point, camera point, proj. reference point)
- Projection plane: in a 3D->2D projection, the plane to which the projection is performed (also view plane)

Projectors

- Projectors: lines from coordinate in Original Space to coordinate in Projected Space



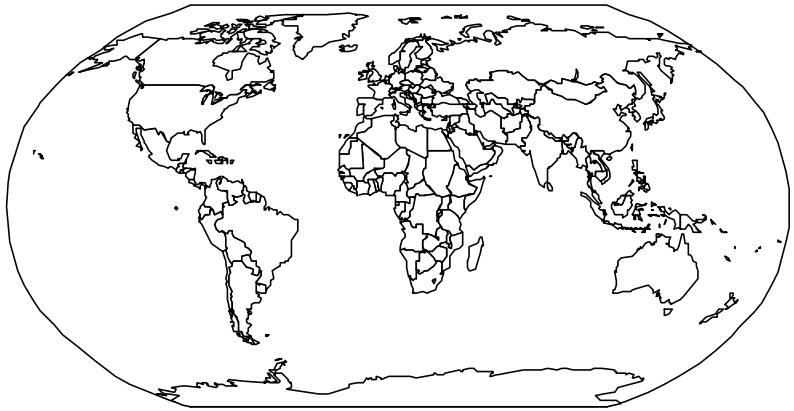
Perspective: Distance to CoP is finite



Parallel: Distance to CoP is infinite

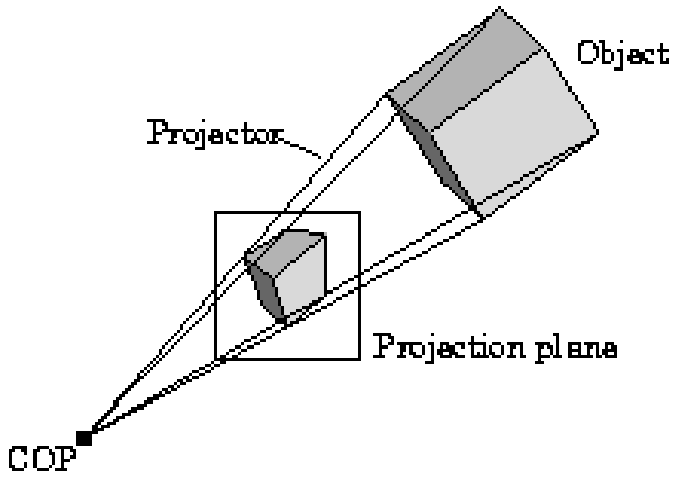
Planar Geometric Projections

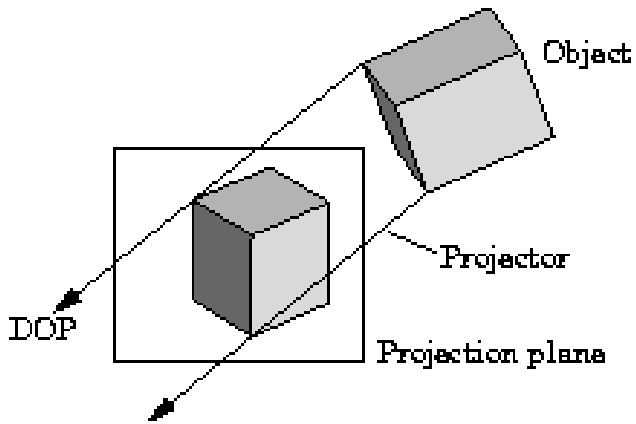
- Projection onto a plane
- Projectors are straight lines
- Alternatives:
  - Some Cartographic Projections
  - Omnimax



Center of Projection

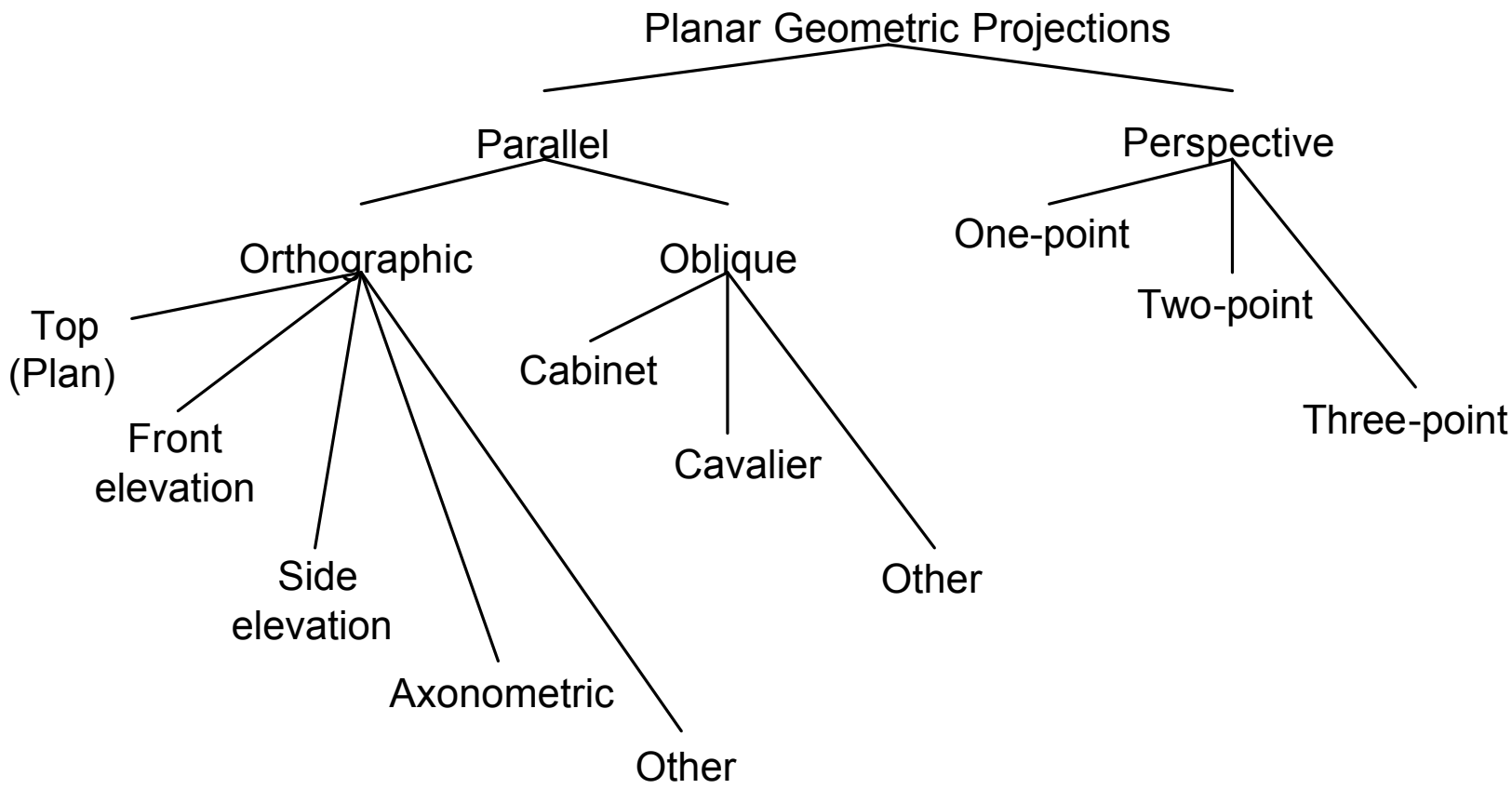
Perspective View





Parallel View

Projections



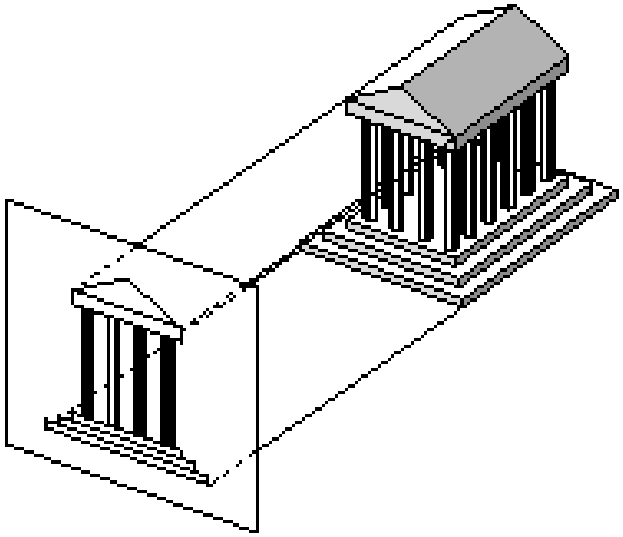
Parallel Projections

- Orthographic: Direction of projection is orthogonal to the projection plane
  - Elevations: Projection plane is perpendicular to a principal axis
    - Front
    - Top (Plan)
    - Side
  - Axonometric: Projection plane is not orthogonal to a principal axis
  - Isometric: Direction of projection makes equal angles with each principal axis.

- Oblique: Direction of projection is not orthogonal to the projection plane; projection plane is normal to a principal axis
  - Cavalier: Direction of projection makes a 45° angle with the projection plane
  - Cabinet: Direction of projection makes a 63.4° angle with the projection plane

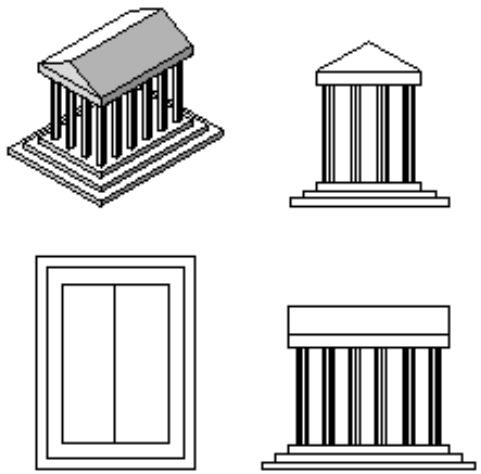
Parallel Projections: Orthographic Projections

- Parallel Projectors Perpendicular to Projection Plane
- Special Case of Perspective Projection



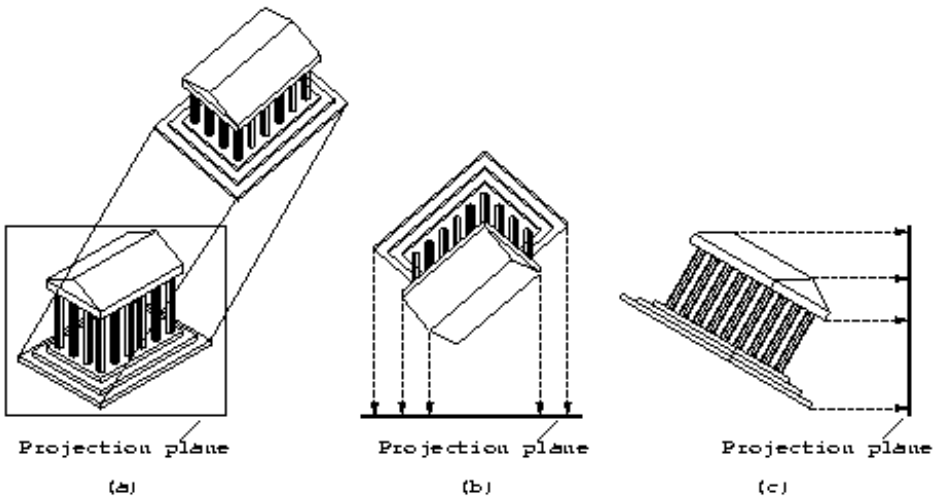
Orthographic Projections: Multiview

- Classical Drafting Views
- Preserves both distance and angles
- Suitable to Object Views, not scenes
- Front-Elevation
- Projection Plane Parallel to Principle Faces
- Top or Plan-Elevation
- Side-Elevation



Orthographic Projections: Axonometric Projections

- Projection plane can have any orientation to object

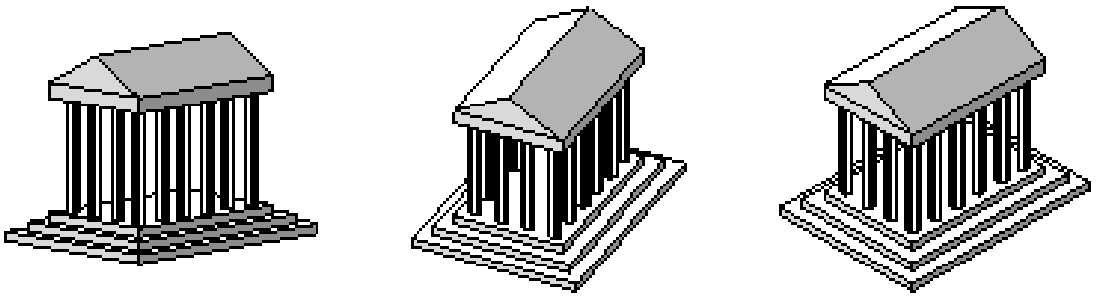


Axonometric Projections

- Isometric
  - Symmetric to three faces
- Dimetric
  - Symmetric to two faces
- Trimetric
  - General Axonometric case

- Foreshortening:
  - Length is shorter in image space than in object space
  - Uniform Foreshortening

  (As opposed to perspective projections where foreshortening is dependent on distance from object to COP)
- Parallel lines preserved
- Angles are not preserved

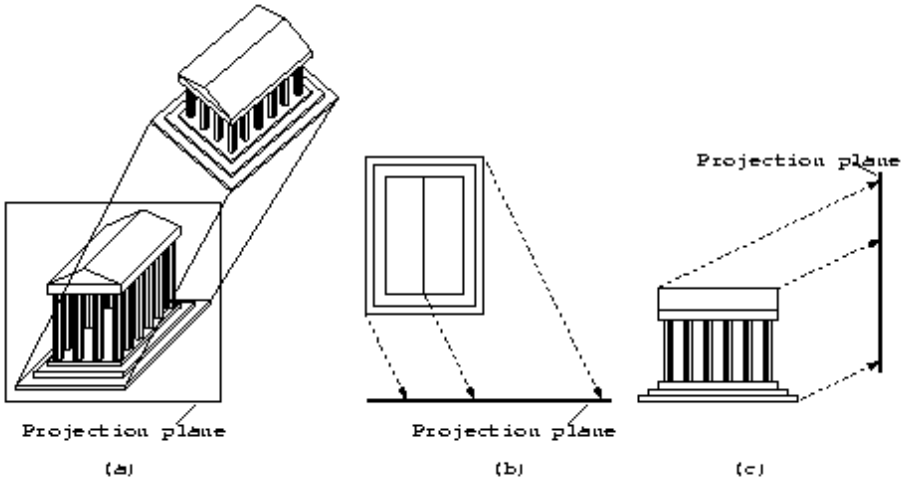


Oblique Projections

- Parallel Projections not perpendicular to projection plane

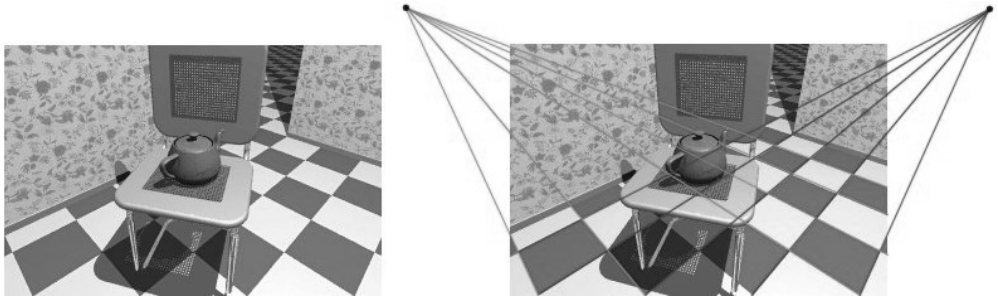
Oblique projection types:

- Cavalier
  - 45-degree Angles from Projection Plane
- Cabinet
  - Arctan(2) or 63.4-degree Angles from Projection Plane



Perspective Projections

- First discovered by Donatello, Brunelleschi, and DaVinci during Renaissance
- Objects closer to viewer look larger
- Parallel lines appear to converge to single point



- One-point:
  - One principal axis cut by projection plane
  - One axis vanishing point
- Two-point:
  - Two principal axes cut by projection plane
  - Two axis vanishing points
- Three-point:
  - Three principal axes cut by projection plane
  - Three axis vanishing points

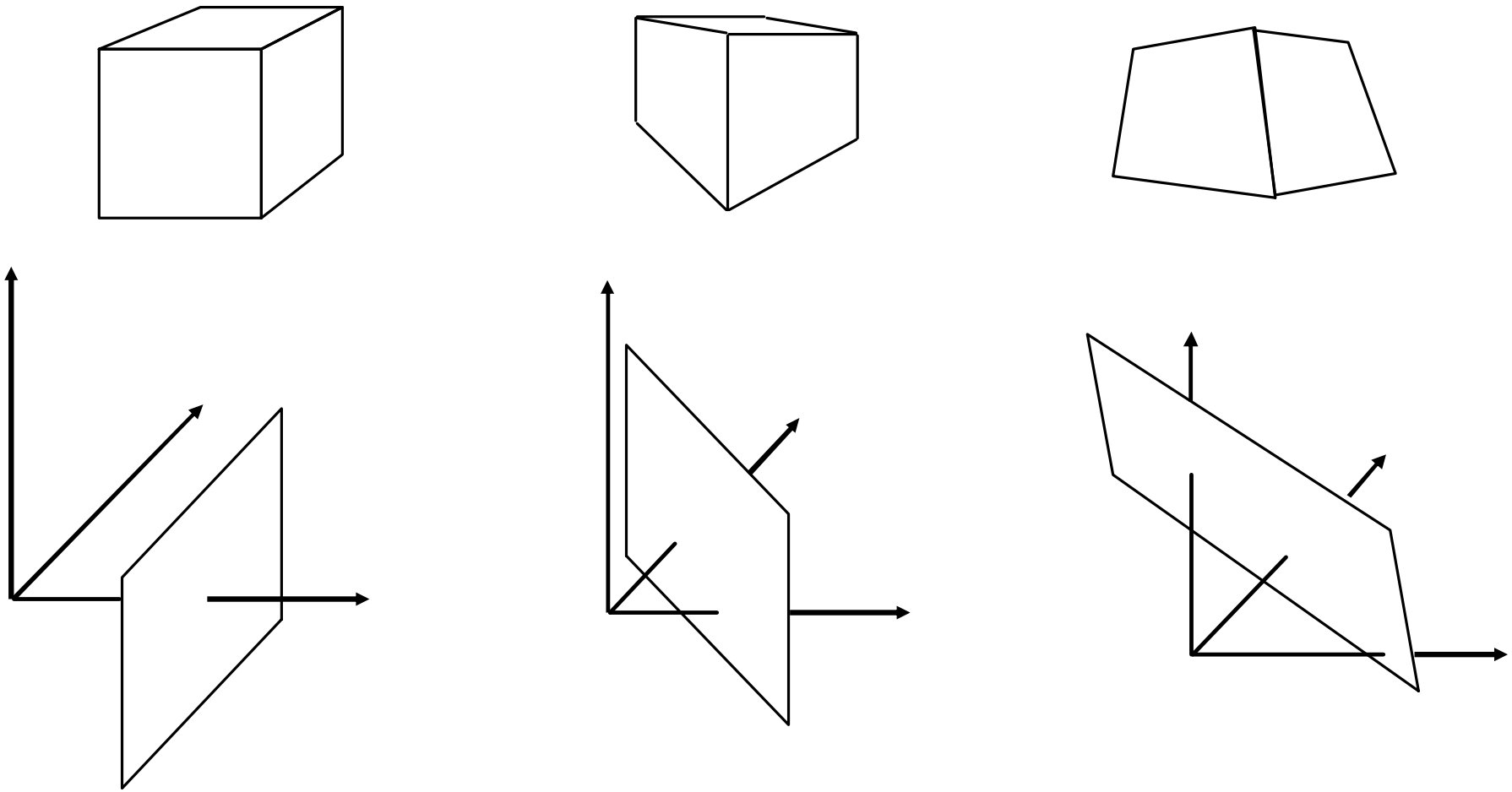
Vanishing Points

- Perspective Projection of any set of parallel line (not perpendicular to the projection plane) converge to a vanishing point
- Infinity of vanishing points
  - one of each set of parallel lines

Axis Vanishing Points

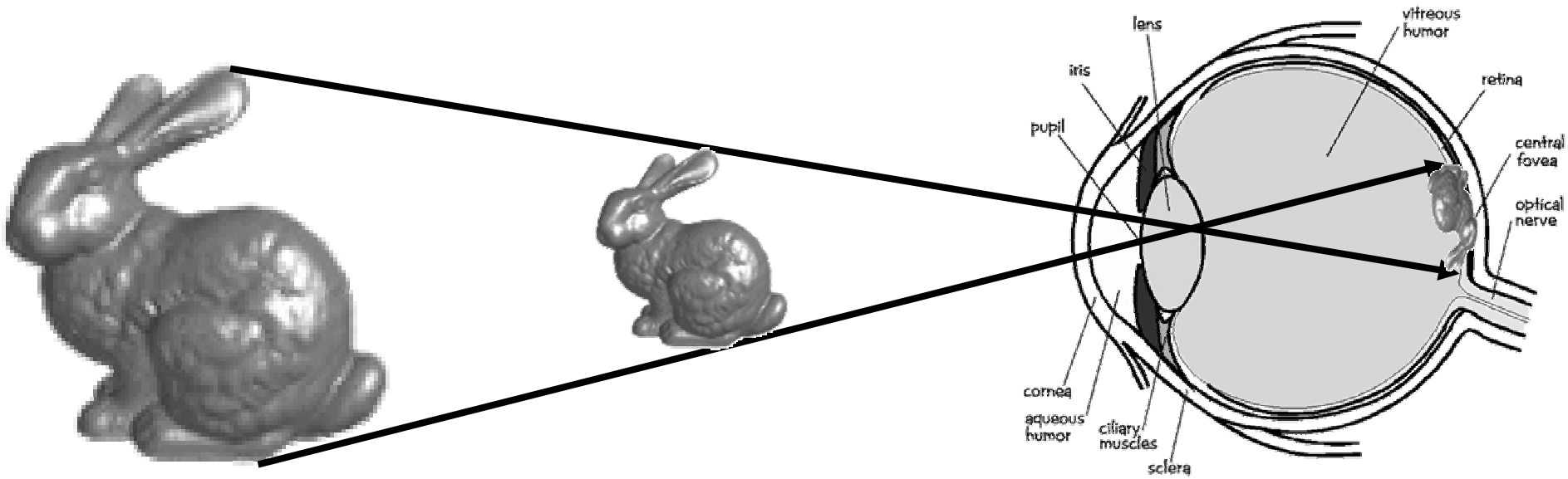
- Vanishing point of lines parallel to one of the three principal axes
- There is one axis vanishing point for each axis cut by the projection plane
- At most, 3 such points
- Perspective Projections are categorized by number of axis vanishing points

Perspective Projections



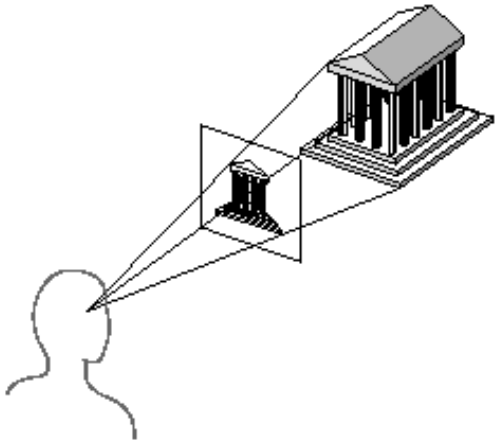
Perspective Projection

- In the real world, objects exhibit perspective foreshortening: distant objects appear smaller
- The basic situation:



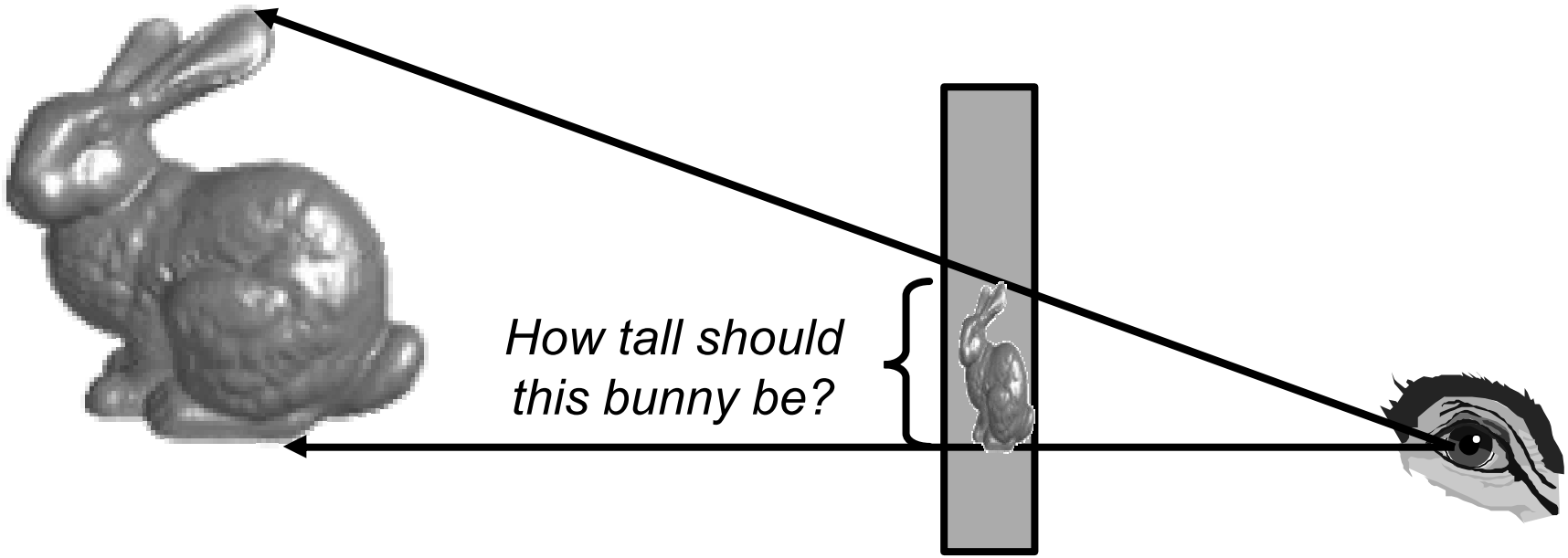
Perspective Views

- Objects further away look smaller
- Natural look
- Length is not preserved
  - Foreshortening of lines depends on distance from viewer



Perspective Projection

- When we do 3-D graphics, we think of the screen as a 2-D window onto the 3-D world:

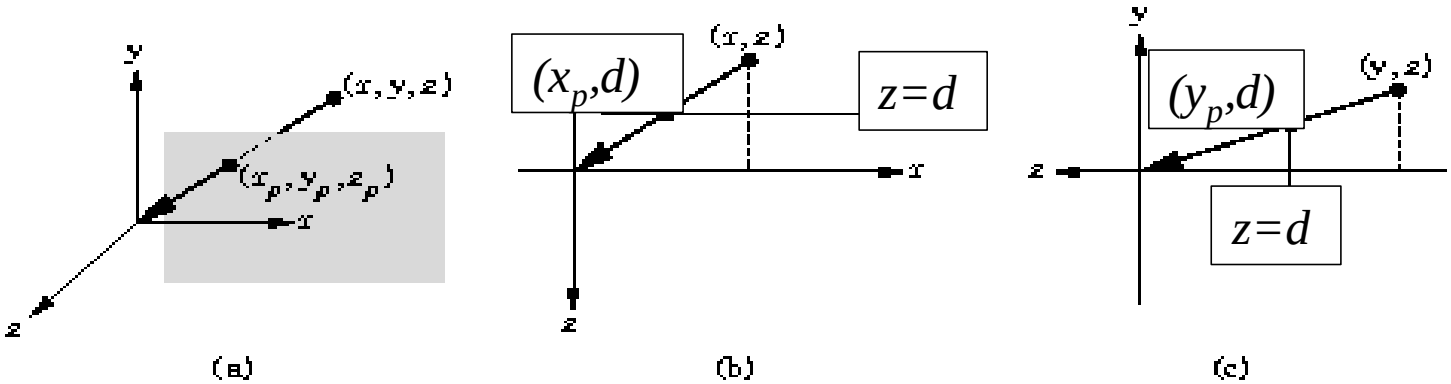


Projection Transformations

NOTE: Throughout the following discussions, we assume an OpenGL-like camera coordinate system (COP at the origin, DOP along the z axis, $z_{vp} < 0$). The concepts are the same for any arbitrary viewing configuration but the math is slightly more complicated.

Projecting Points onto a Display Plane*

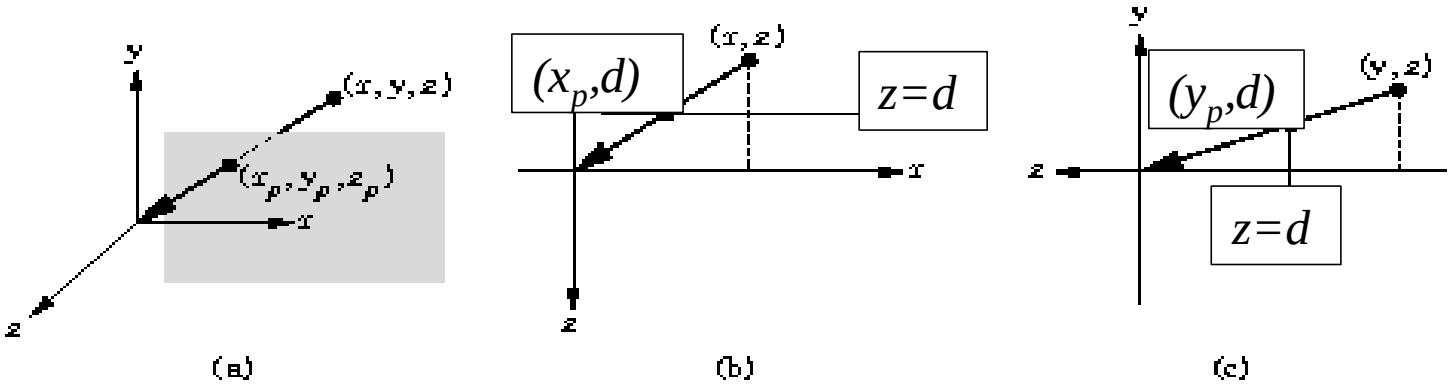
- Projection of world onto display plane involves a perspective transformation:
  - $p' = M_{per} p$
  - Not affine (parallel lines do not remain parallel)
  - Non-reversable



*Note: Angels book has several errors in the mathematics in this area

Projection of point onto display plane

- Observe: $\frac{x}{z} = \frac{x_p}{d}, \quad x_p = \frac{x}{z/d}$
- (Same for y)
- Results in non-uniform foreshortening

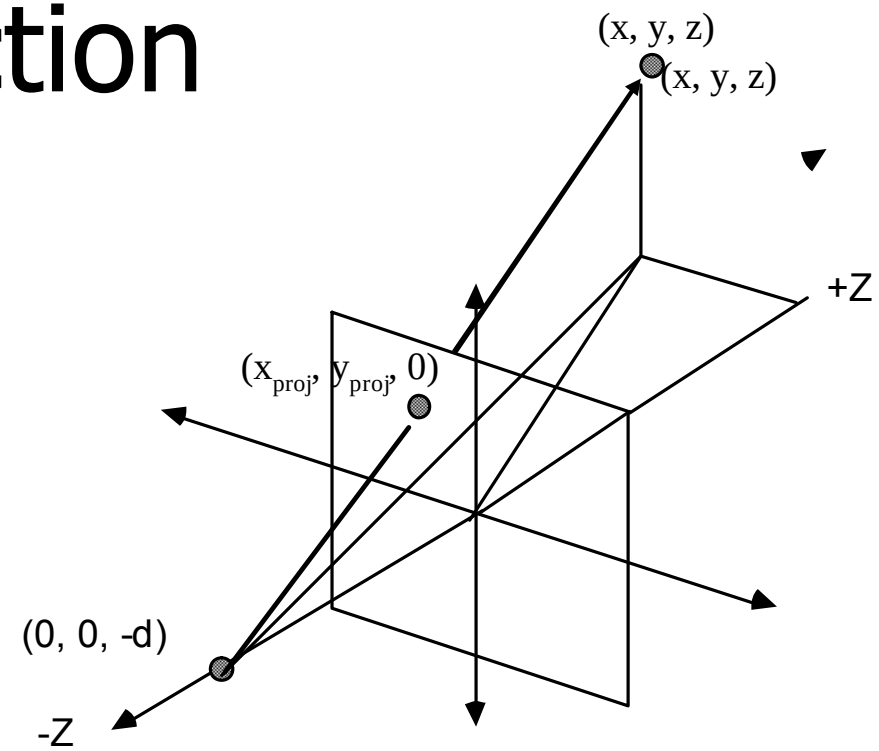


One-Point Projection

Center of Projection on the negative z-axis with viewplane in the x-y plane.

$\mathbf{x_{projected} = xd/(d+z) = x/(1+(z/d))}$

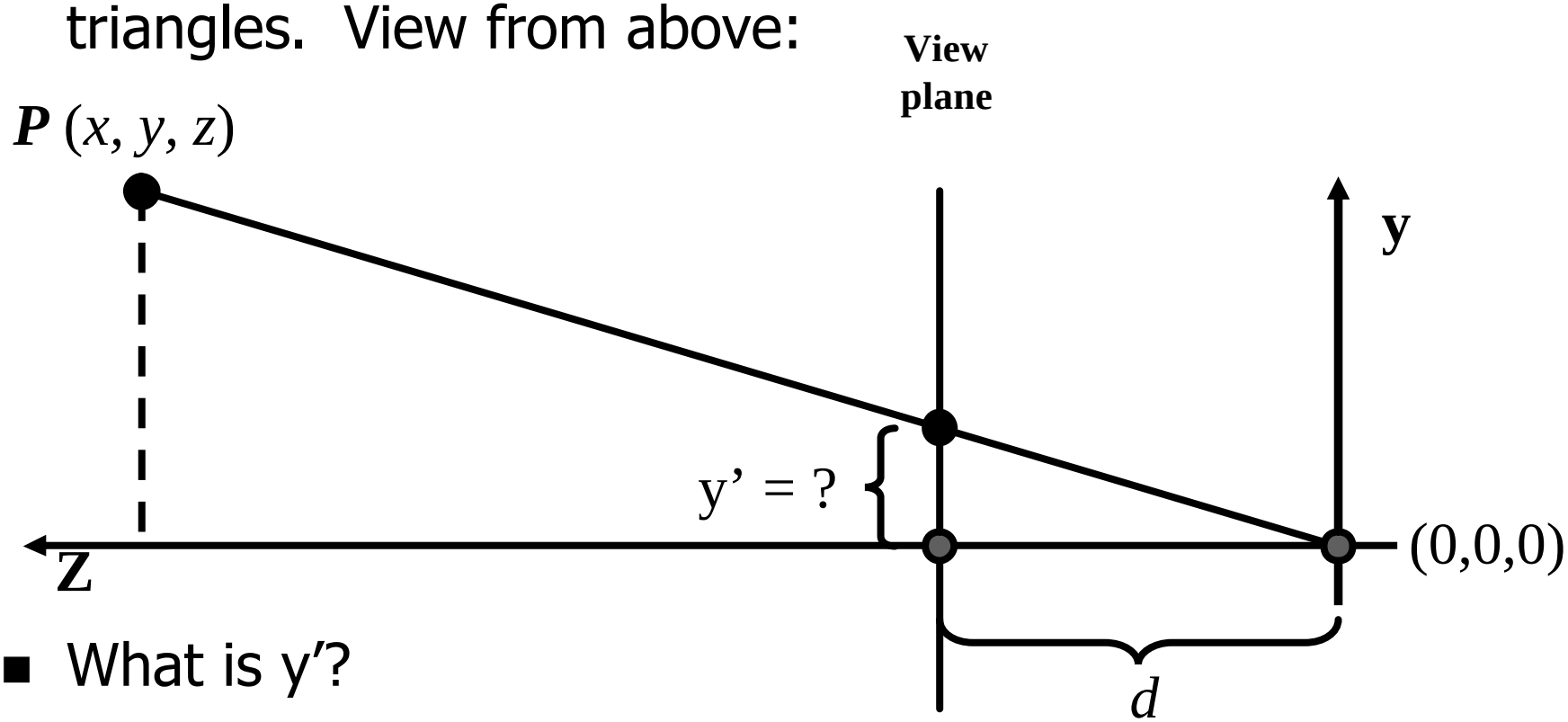
$\mathbf{y_{projected} = yd/(d+z) = y/(1+(z/d))}$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1+(z/d) \end{bmatrix} = \begin{bmatrix} x/(1+(z/d)) \\ y/(1+(z/d)) \\ 0 \\ 1 \end{bmatrix}$$

Perspective Projection step by step

- The geometry of the situation is that of similar triangles. View from above:



- What is y'?

Perspective Projection

- Desired result for a point $[x, y, z, 1]^T$ projected onto the view plane:

$$\frac{x'}{d} = \frac{x}{z}, \quad \frac{y'}{d} = \frac{y}{z}$$

$$x' = \frac{d \cdot x}{z} = \frac{x}{z/d}, \quad y' = \frac{d \cdot y}{z} = \frac{y}{z/d}, \quad z = d$$

- What could a matrix look like to do this?

Use of Homogeneous Coordinates

- We can use homogeneous coordinates to make perspective transformation easier
- Homogeneous Representation of point: $p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$
- Suppose, instead: $p = \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix}$
- As long as $w \neq 0$, we can recover original point

Use of w in homogeneous coordinates

- Let matrix M transforms point p into point q:

$$q = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix} = Mp = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

What is q?

- Then convert q back to a 3D point:

$$q' = \begin{bmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ \frac{z}{z/d} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ d \\ 1 \end{bmatrix} = \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix}$$

- **Thus q' is our projection of p onto the display plane!**

A Perspective Projection Matrix

- Example:

$$\begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Or, in 3-D coordinates:

$$\left(\frac{x}{z/d}, \quad \frac{y}{z/d}, \quad d \right)$$

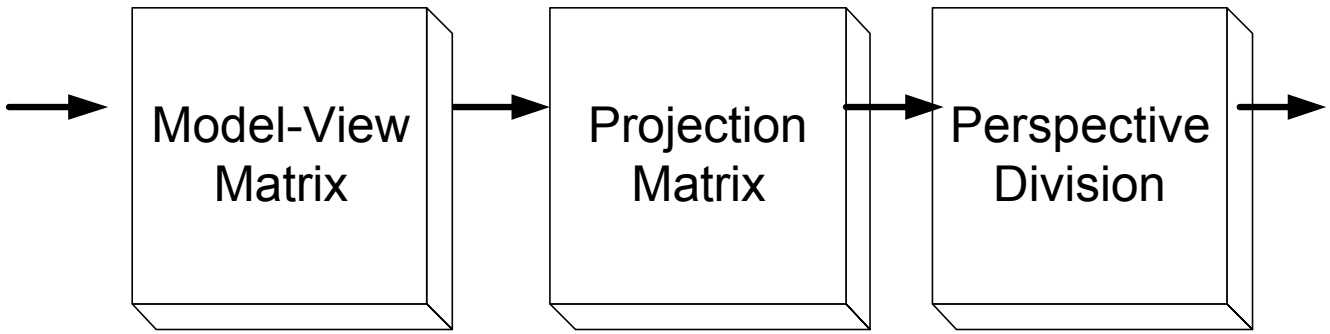
Thus, M is M_{pers}

- Thus, the matrix M is used to project points onto a perspective display plane

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$

Simple Projection Pipeline

- Simple OpenGL-like Projection Matrix:



Orthogonal Projections, M_{ortho}

- Special Case of Perspective Projection
- Display Plane at z=0

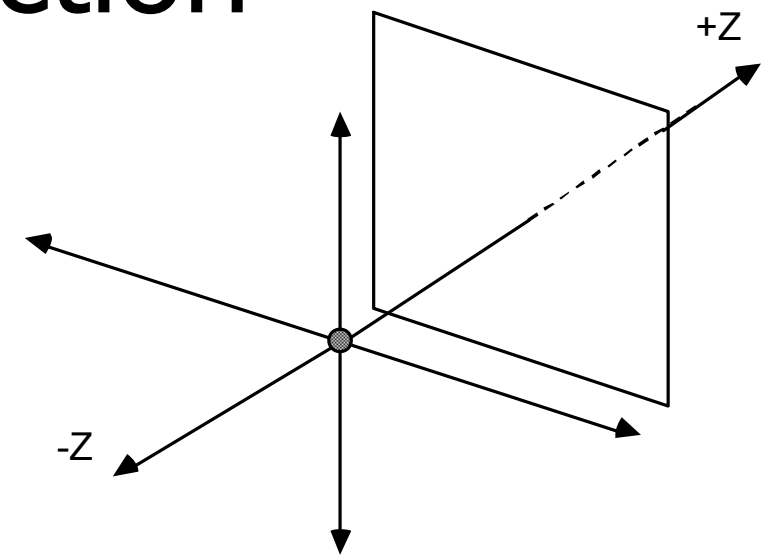
$$x_p = x,$$

$$y_p = y,$$

$$z_p = 0$$

$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Another One-Point Projection



Center of Projection at the origin with view plane parallel to the x-y plane at distance d from the origin.

$\mathbf{x_{projected} = dx/z = x/(z/d)}$

$\mathbf{y_{projected} = dy/z = y/(z/d)}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix} \longrightarrow \begin{bmatrix} x/(z/d) \\ y/(z/d) \\ d \\ 1 \end{bmatrix}$$

Perspective division

$\mathbf{M_{per}}$

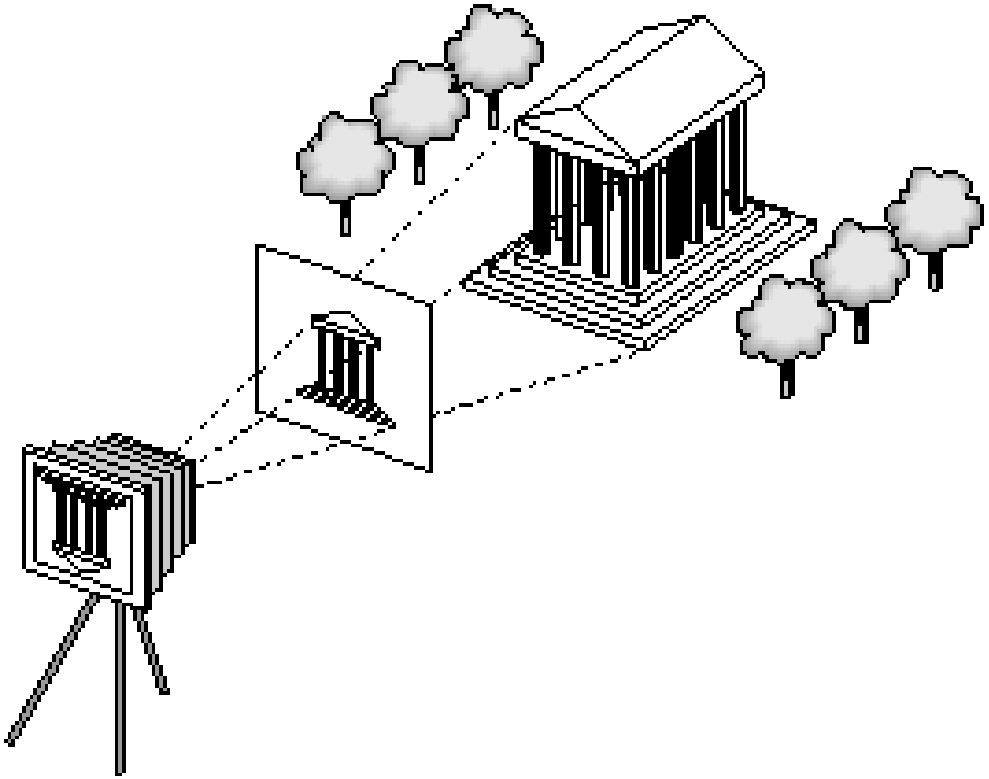
Points plotted are x/w, y/w where w = z/d

Specifying An Arbitrary 3-D View

- Two coordinate systems
  - World reference coordinate system (WRC)
  - Viewing reference coordinate system (VRC)
- First specify a view plane and coordinate system (WRC)
  - View Reference Point (VRP)
  - View Plane Normal (VPN)
  - View Up Vector (VUP)
- Specify a window on the view plane (VRC)
  - Max and min u,v values (Center of the window (CW))
  - Projection Reference Point (PRP)
  - Front (F) and back (B) clipping planes (hither and yon)

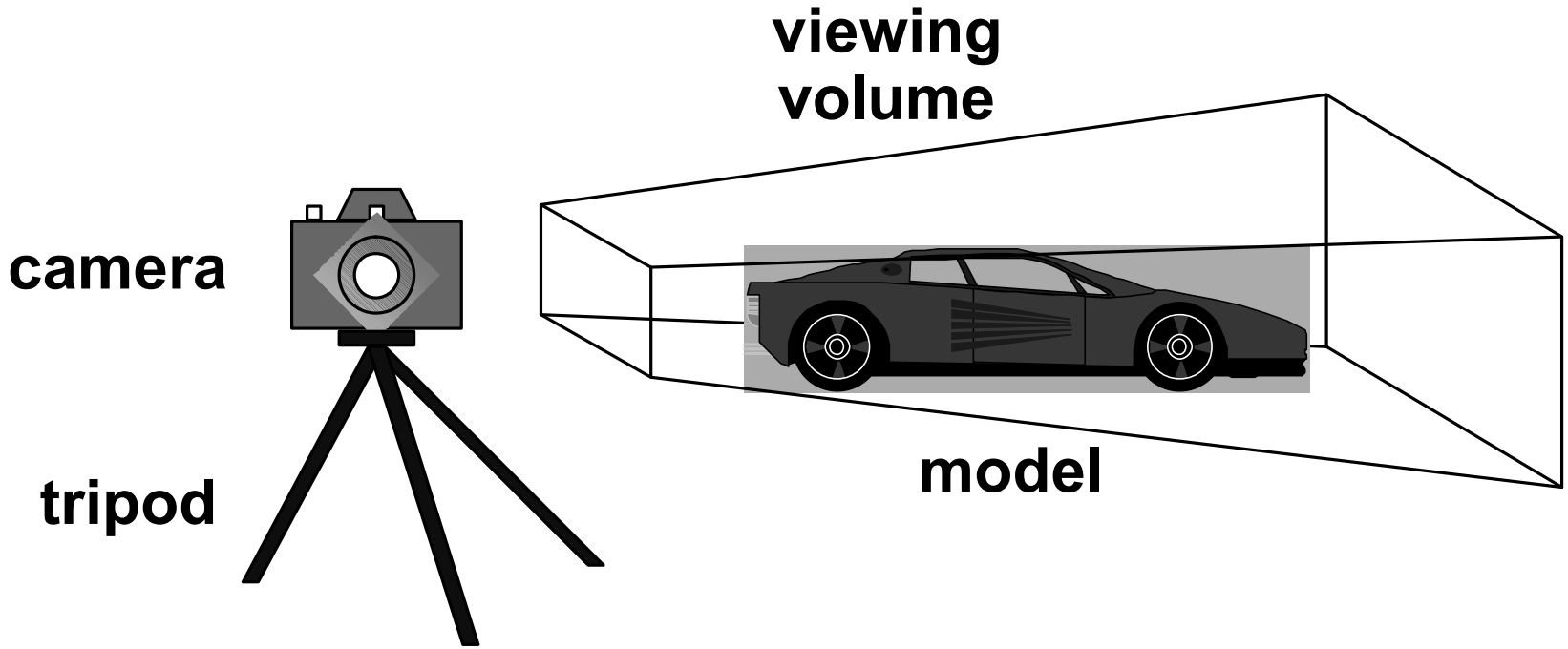
Synthetic Camera Model

Synthetic Camera Model

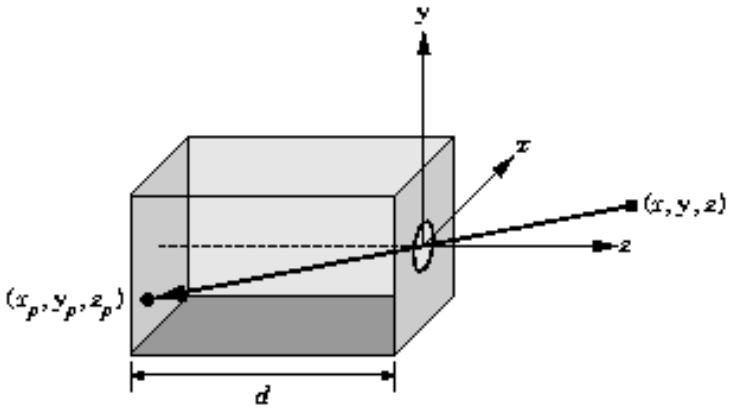


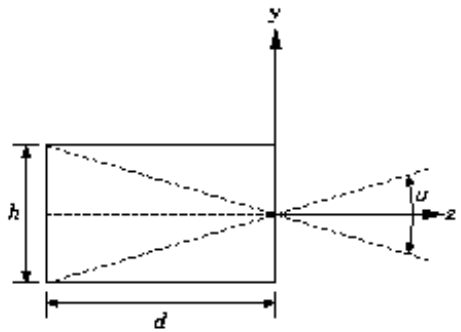
Camera Analogy

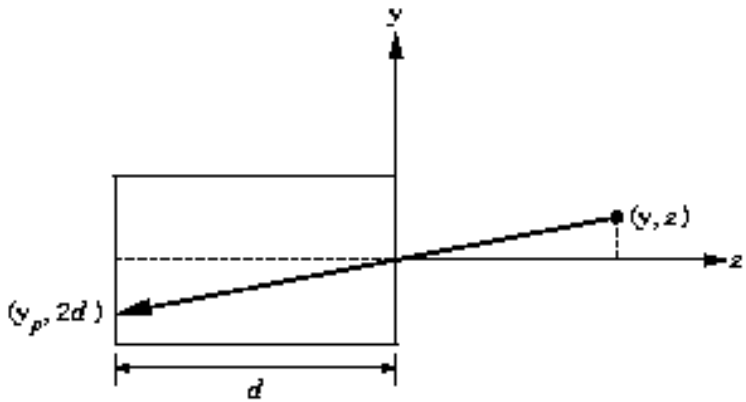
- 3D is just like taking a photograph (lots of photographs!)



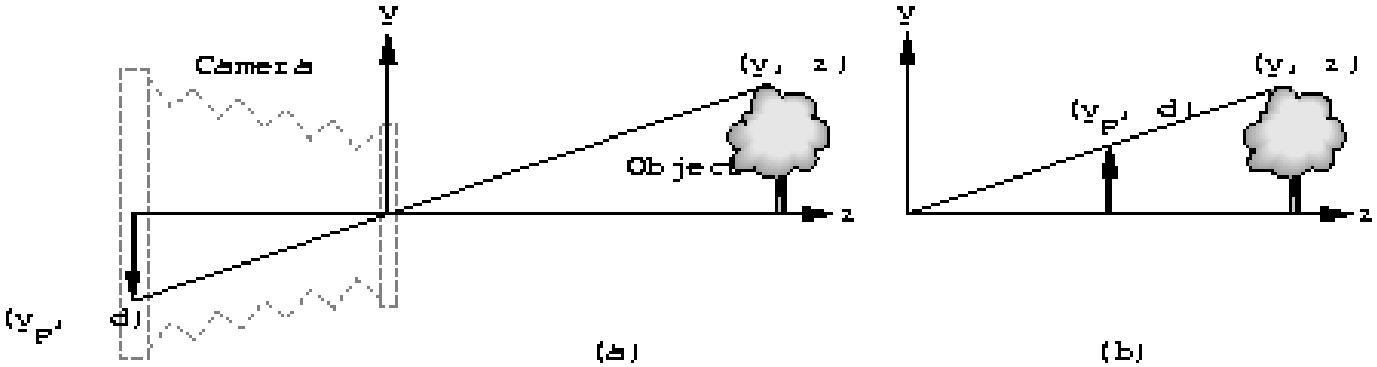
Camera







Camera Projection



Camera Analogy and Transformations

- Projection transformations
  - adjust the lens of the camera
- Viewing transformations
  - tripod–define position and orientation of the viewing volume in the world
- Modeling transformations
  - moving the model
- Viewport transformations
  - enlarge or reduce the physical photograph

Coordinate Systems and Transformations

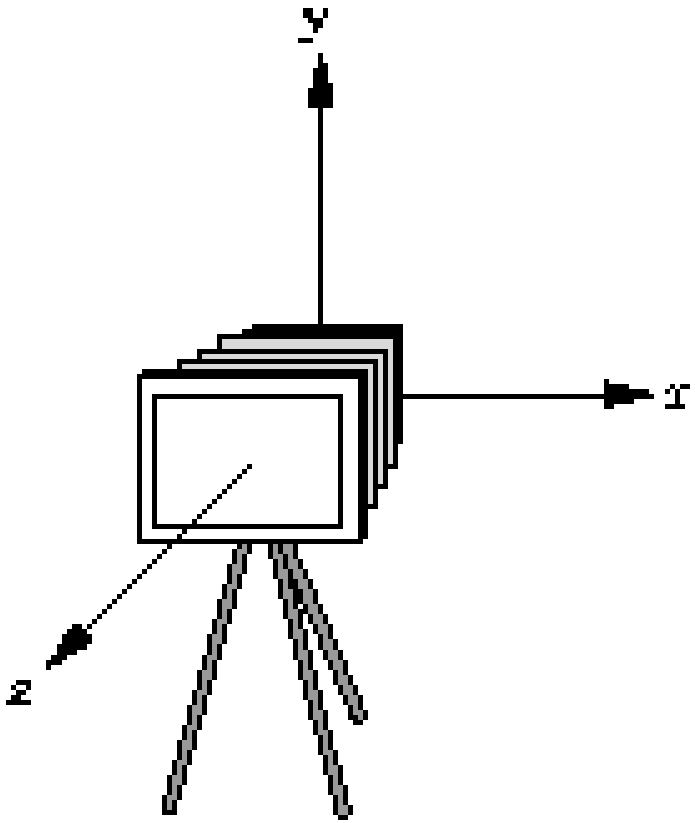
- Steps in Forming an Image
  - specify geometry (world coordinates)
  - specify camera (camera coordinates)
  - project (window coordinates)
  - map to viewport (screen coordinates)
- Each step uses transformations
- Every transformation is equivalent to a change in coordinate systems (frames)

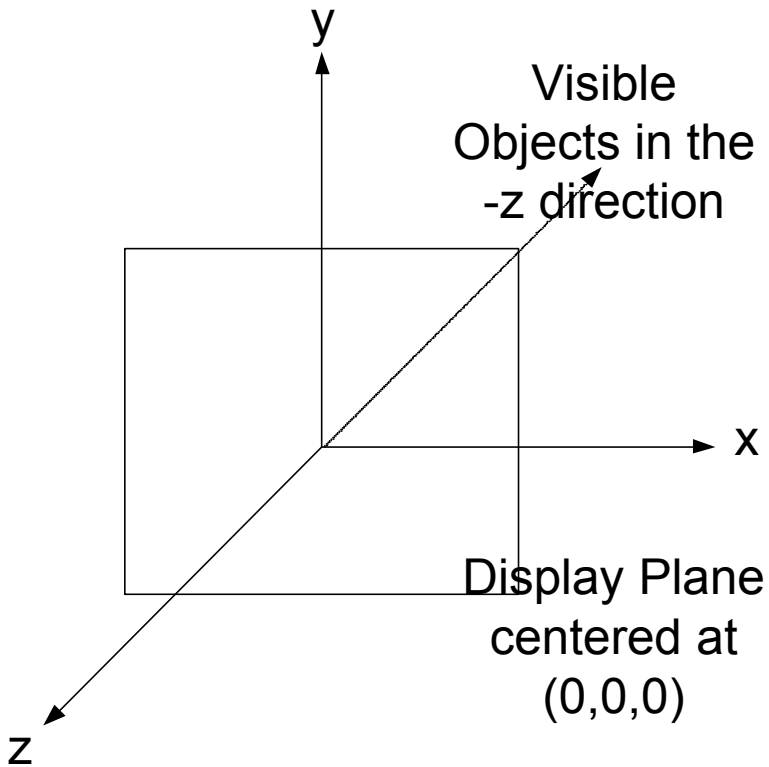
Projection Specification

Projection Characteristics

- Camera/Viewpoint Location
- Camera/Viewpoint Direction
- Camera/Viewpoint Orientation
- Camera/Viewpoint Lens (View Volume)
  - Width/Height of Lens
  - Front/Back Clipping Planes
- Parallel or Perspective Projections
  - Parallel is special case of Perspective

OpenGL Camera or Projection Coordinates





Specifying Viewing Characteristics

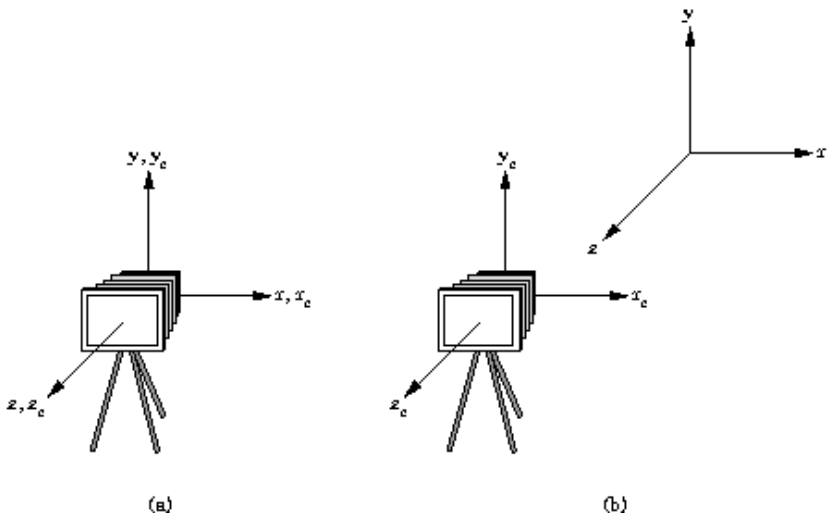
- Fixed Camera / Move World
- “Look At”
- View Volume / Display Plane Specification
- Vector Specification

Camera Location Relative to World Coordinates

- Can be thought of in two ways:
  - Camera location is specified in world coordinates
  - World coordinate frame is located in camera coordinates
- Camera transformations are reverse of World transformations

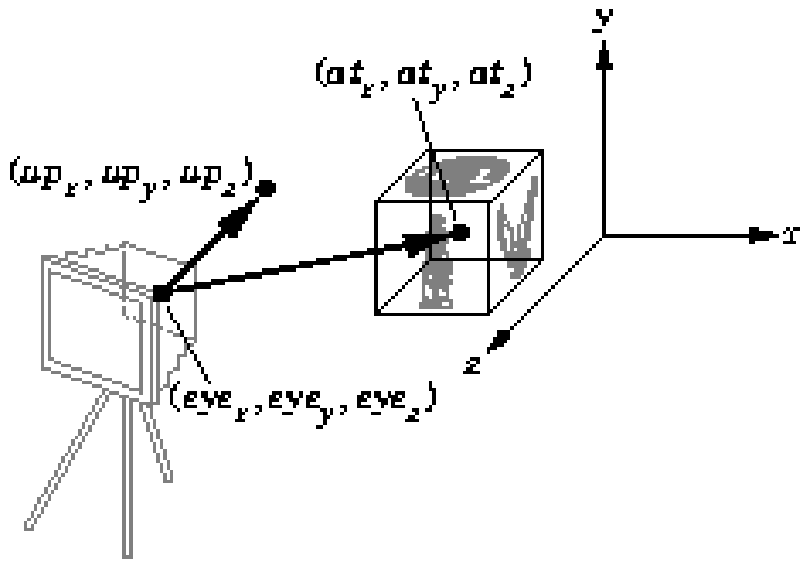
Fixed Camera / Move World

- Fix the camera at a specific location/orientation
- Transform the world such that the camera sees the world the "right" way – I.e., move the world, not the camera
- Typical approach for OpenGL



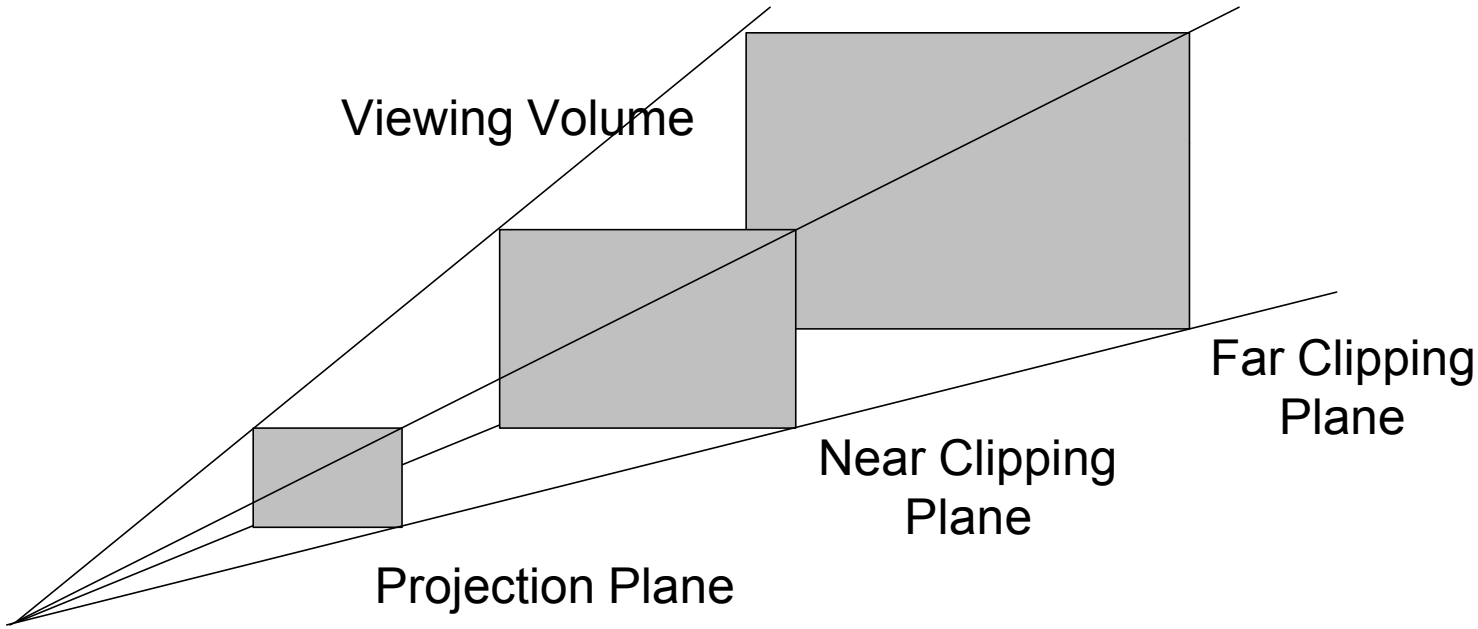
“Look At”

- Setup the camera to “Look At” the world
- Set the camera location
- Set the “look at” point
- Set the camera rotation angle
- Example: gluLookAt



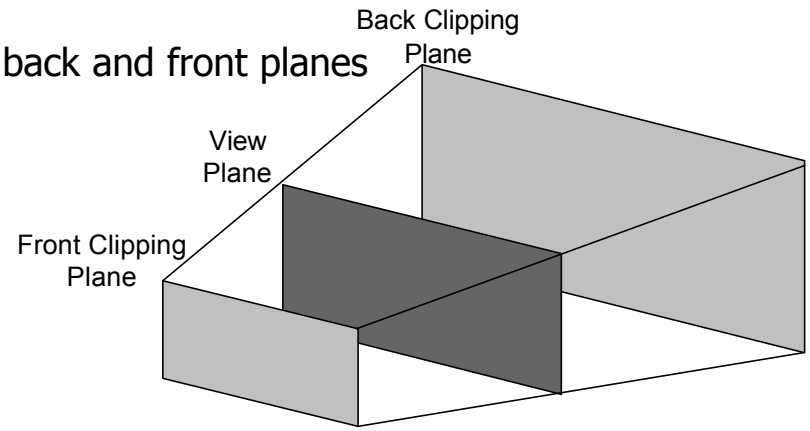
View Volume / Display Plane Specification

- Specify View Volume and Display Plane



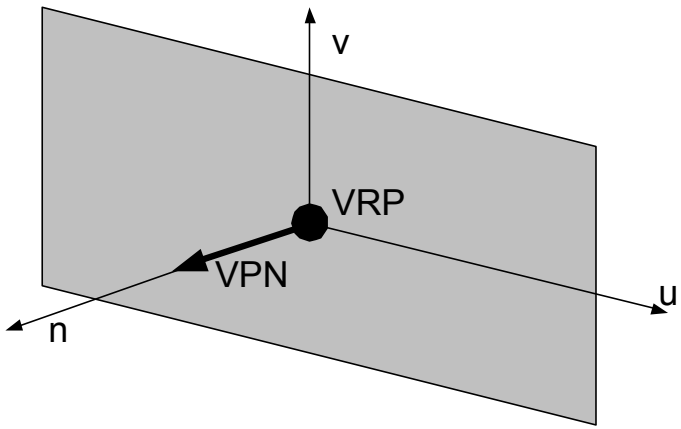
Fun with View Volumes

- All projection information is captured in view volume and projection plane.
- What happens if we play with these values?
  - Oblique Parallel Projections
  - Non-Right Frustum Perspective Projections
- View Volume Values
  - Box
    - Lower-Left Corner/Upper Right Corner of back and front planes
    - Back Plane Angles/Front Plane Angles
  - Projection Plane
    - Lower-Left Corner/Upper-Right Corner
    - Angle
    - Location Relative to View Volume



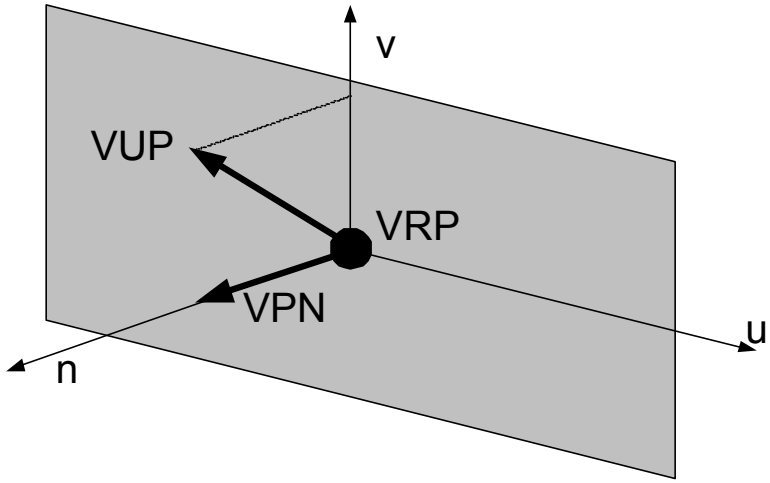
Vector Specification

- Most flexible method of viewing specification
- View plane may be anywhere with respect to the world
- Locate the View Plane (I.e. Projection Plane) by:
  - View reference point (VRP)
  - View-plane normal (VPN) (n axis)



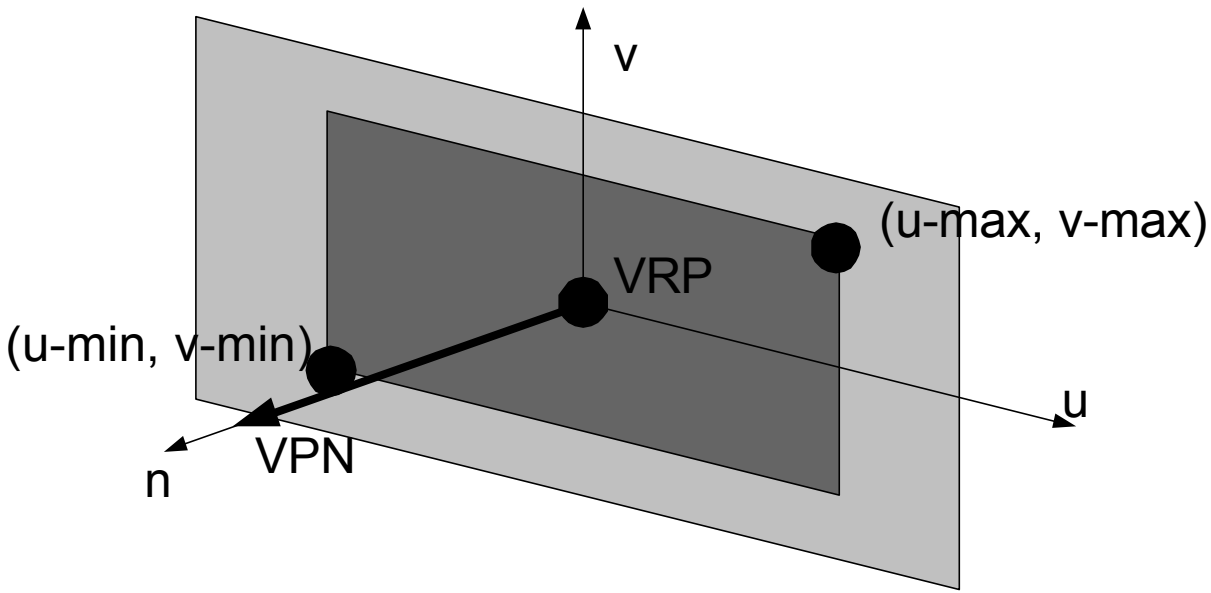
VRC Coordinate System

- Viewing-Reference Coordinate (VRC) System
  - n-axis - along View-plane normal (VPN)
  - v-axis – projection of View-up Vector (VUP)
  - u-axis – Right hand coordinate system



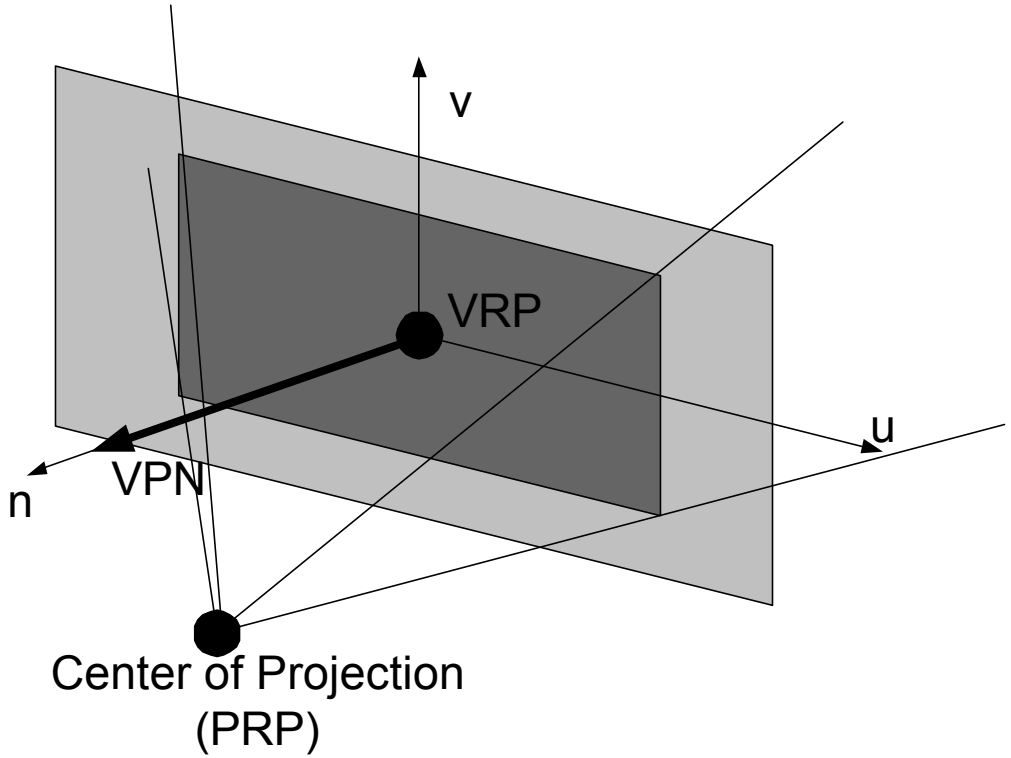
Viewing Window

- Min/Max u and v ranges
- Need not be symmetrical around VRP



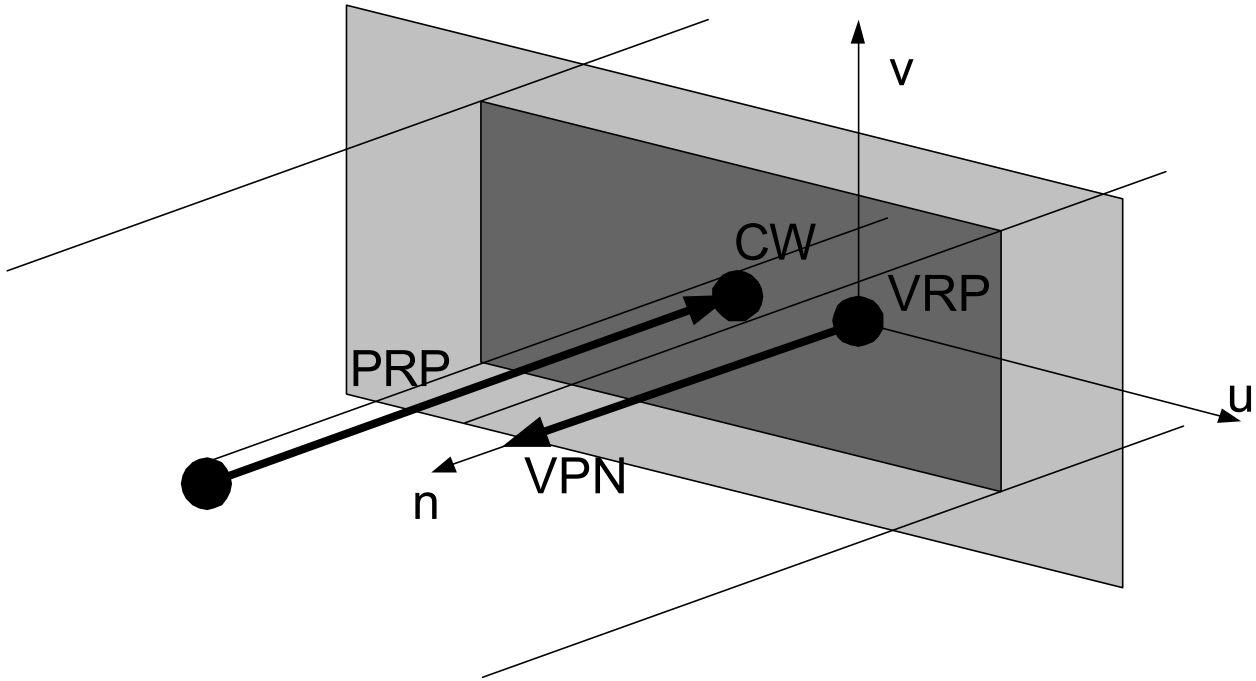
Center of Projection

- Projection-Reference Point (PRP)



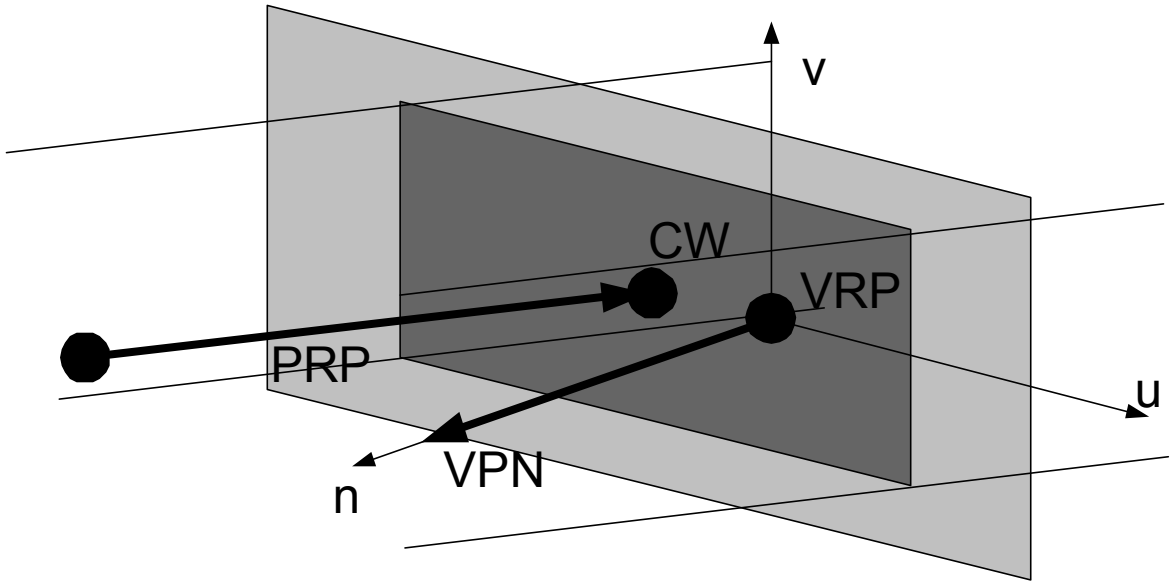
Direction of Projection

- Projection-Reference Point (PRP)

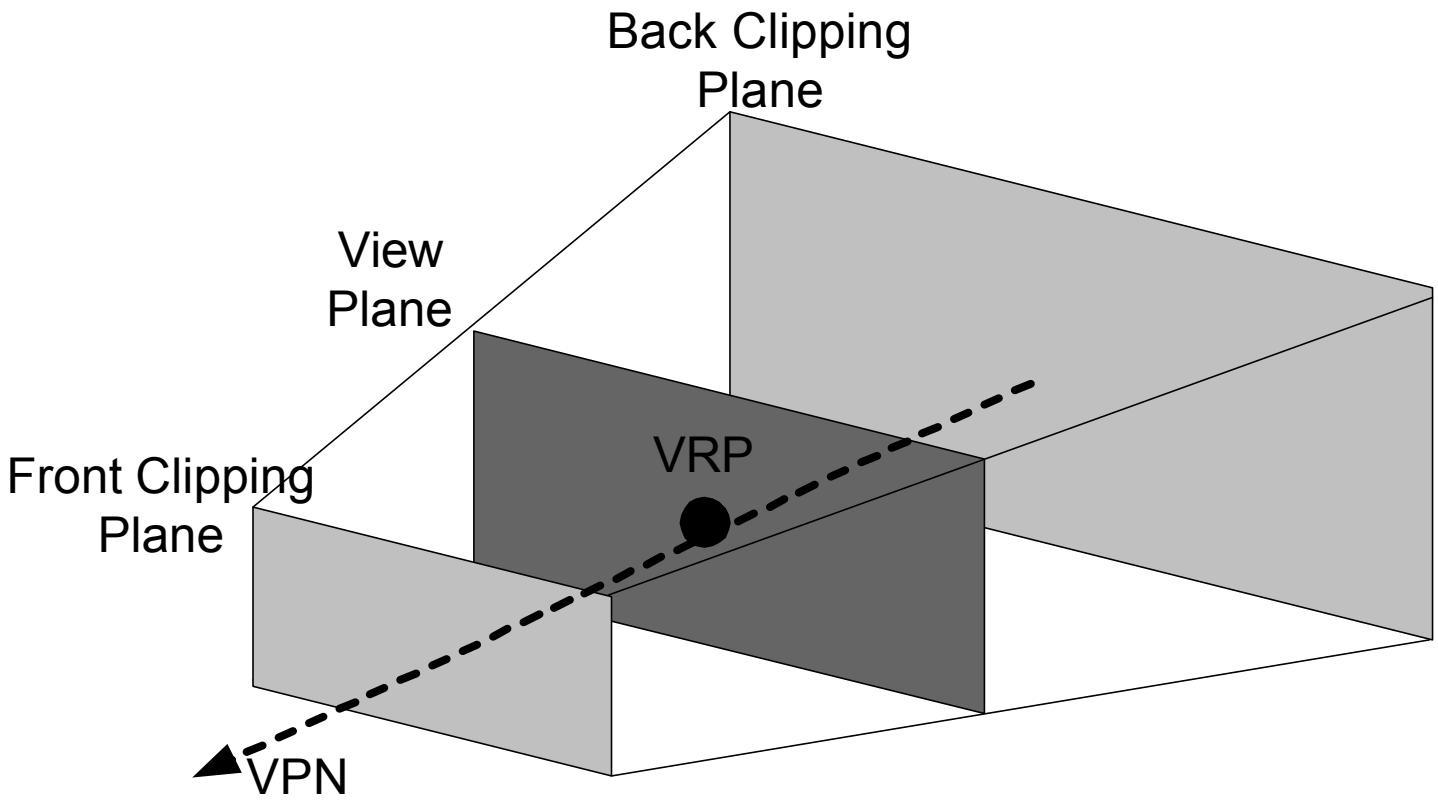


Oblique Parallel Projection

- Vector from PRP to Center of the Window not parallel to VPN

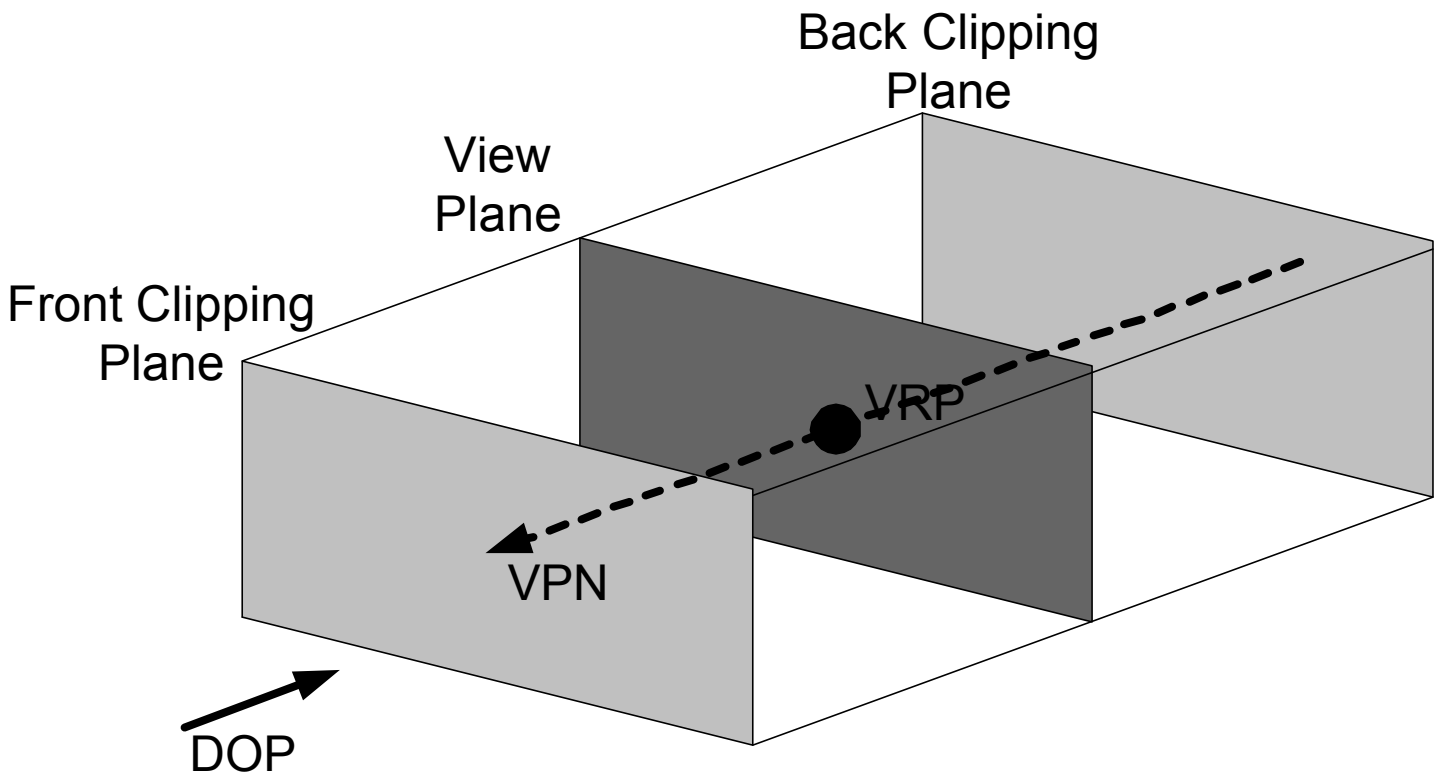


Perspective View Volumes

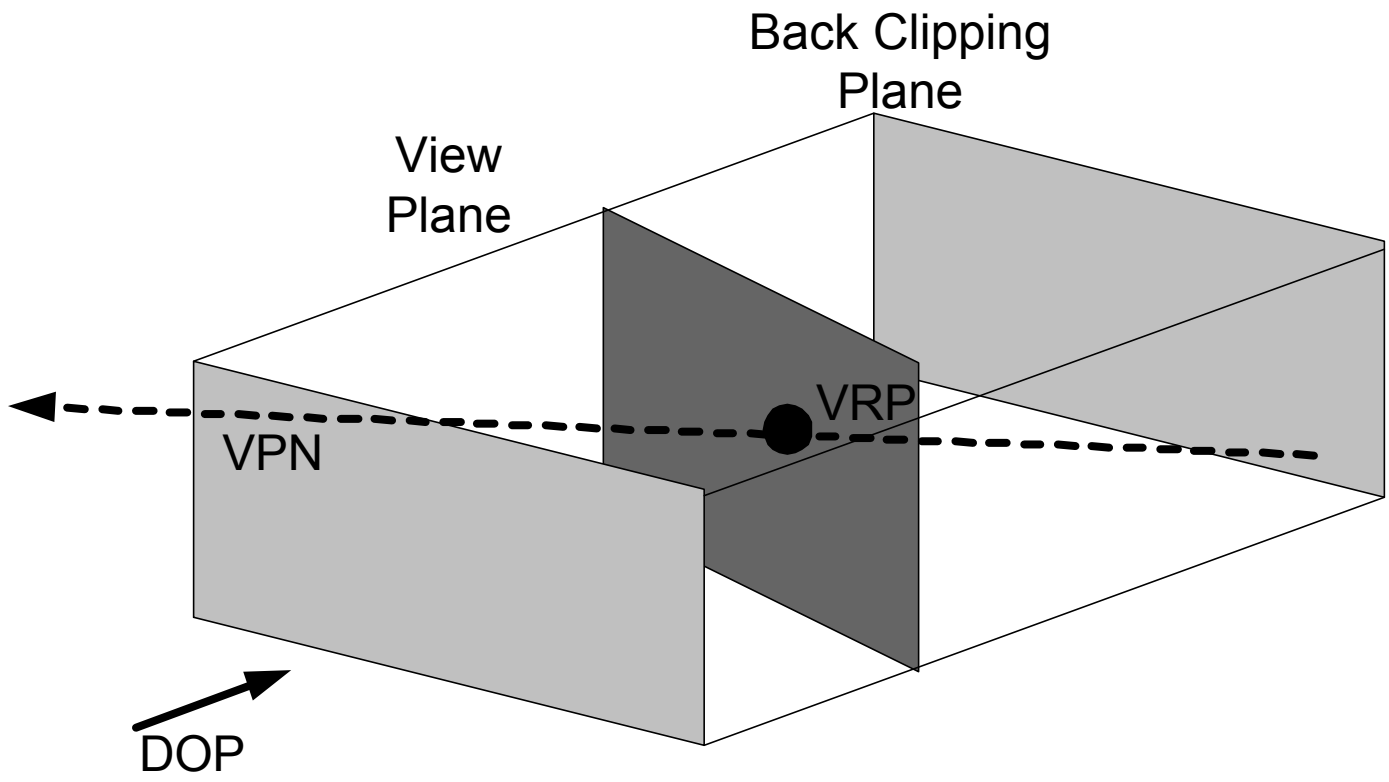


Orthogonal View Volumes

- Specification of Viewing Volume



Oblique Projections



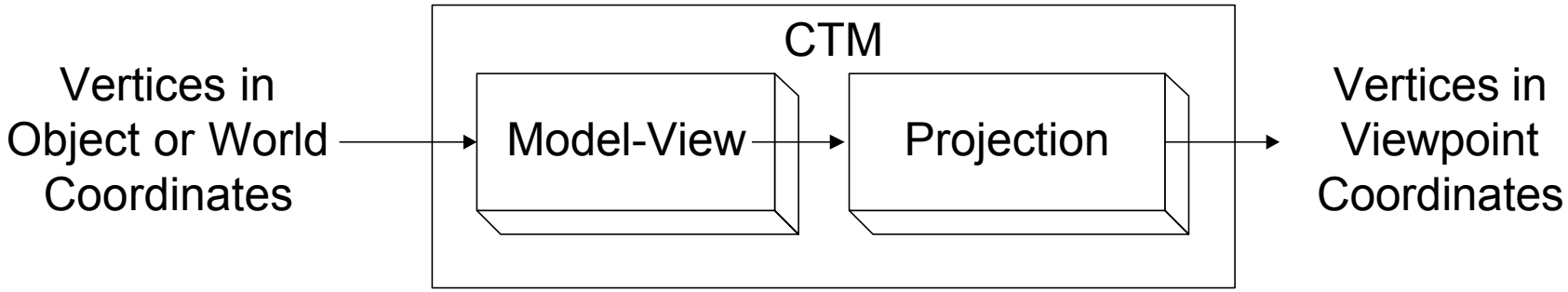
Projection Specification in OpenGL

Projection Specification in OpenGL

- Locate World in Camera Coordinates
  - Alternative: **`gluLookAt`** encapsulates world to camera coordinate transformations
- Select View Type (Pespective or Orthogonal)
- Set View Volume (**`glFrustum`**, **`glOrtho`**, **`gluPerspective`**)
  - (near clipping plane is projection plane)
- Conceptually not as flexible as VRC system
  - Although with world coordinate transformation, you can obtain the same results

Projections in OpenGL

- Camera located at (0,0,0) pointing in the –z direction. Up is positive y direction.
- **`GL_Modelview`** Matrix controls the world
- **`GL_Projection`** Matrix controls the camera

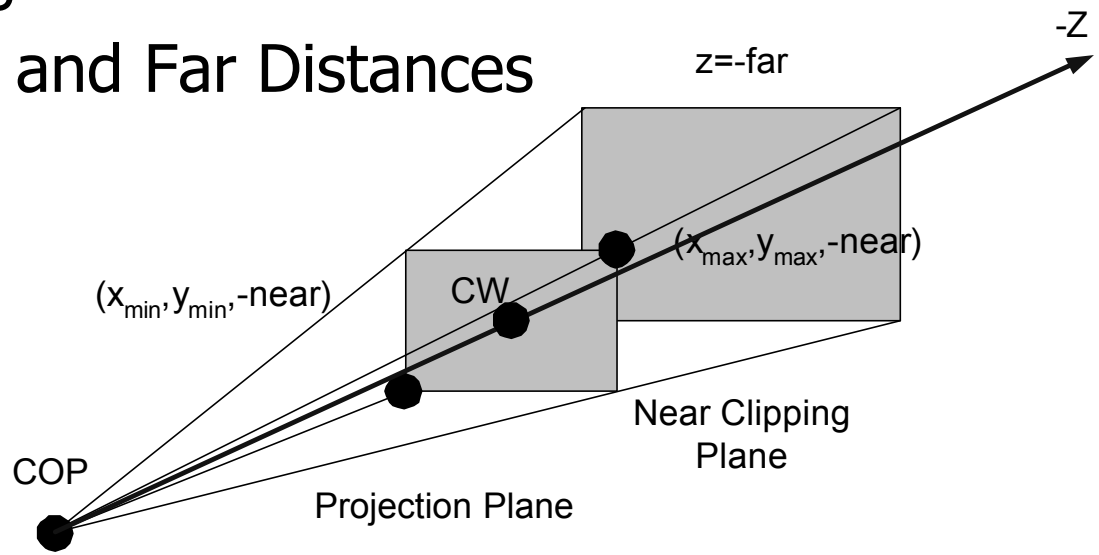


OpenGL

- Move the world relative to the camera
- For example:
  - Moving the camera +10 along the Z axis is equivalent to moving the world –10 along the Z axis
  - Is rotating the camera positively around the y axis the same as rotating the world negatively around the y axis?

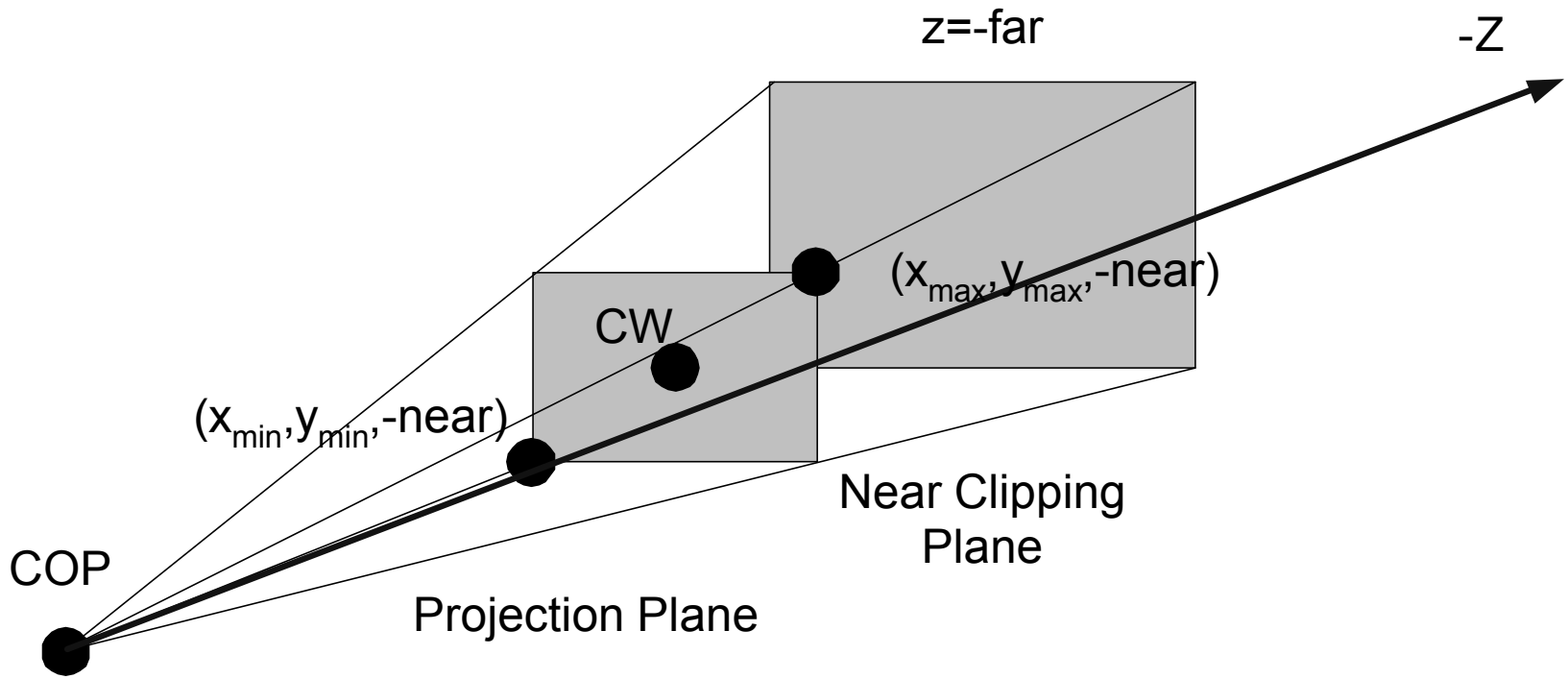
OpenGL Implementation of View Volume / Display Plane

- glFrustum
  - Display Plane is same as Near Clipping Plane
  - Specify Lower-Left and Upper Right Corners of Near Clipping Plane
  - Specify Near and Far Distances



glFrustum Variations

- Display Axis (-z) does not have to go through the center of the window



OpenGL Alternatives

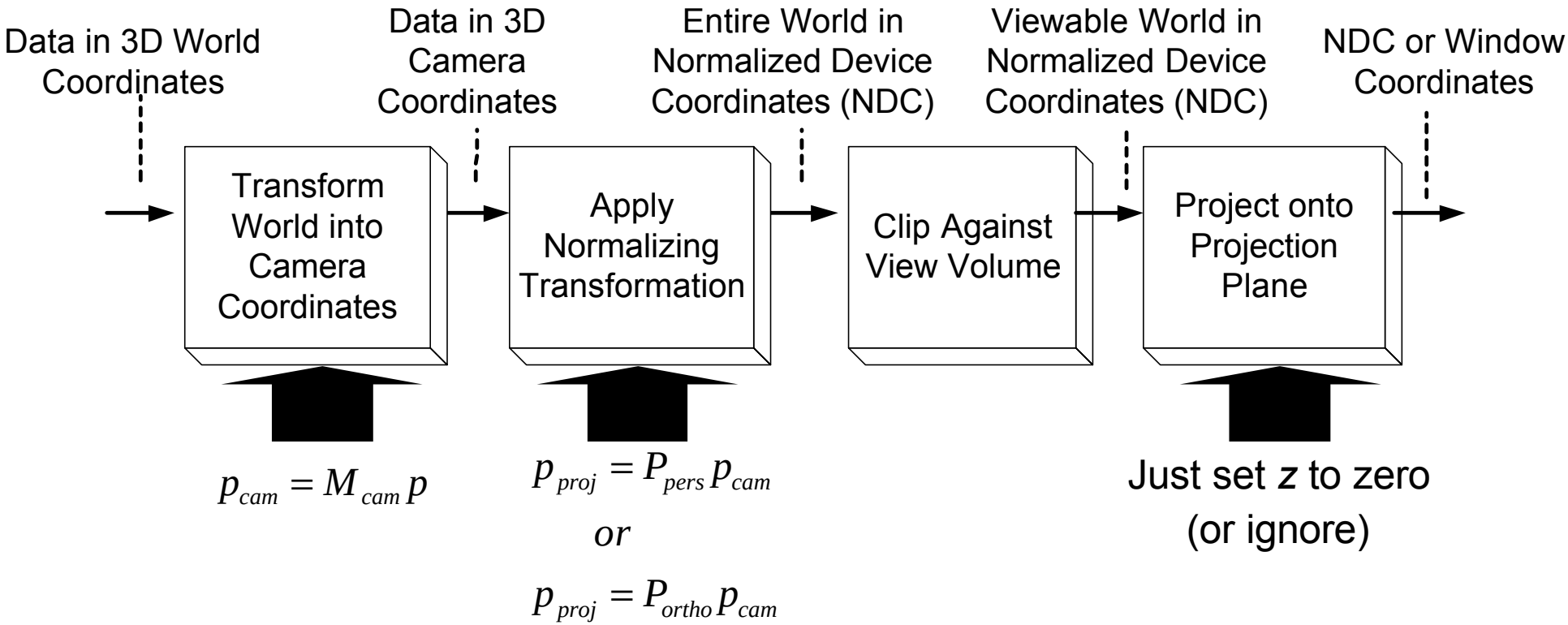
- Alternative: gluPerspective
  - Specifies Frustum via viewing angles
  - (ala Camera Lens)
- glOrtho – Orthogonal Projection

Now What?

- Several issues are not address with the simple projection matrixes we have developed:
  - 3D Clipping Efficiency in a Frustum Viewing Volume
  - Hidden Surface Efficiency
- Solution: Use Projection Normalization
  - Get rid of perspective and other problem projections!
  - Everything is easier in an canonical, orthogonal world!

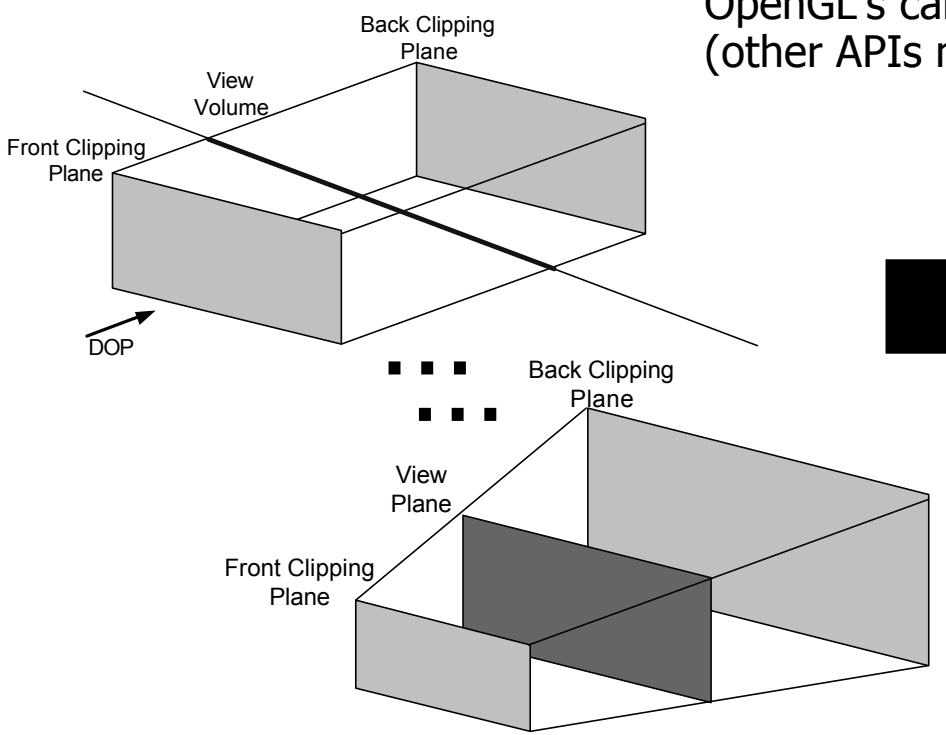
3D Viewing Transformation Projection Matrixes

- Input 3D World Coordinates
- Output 3D Normalized Device Coordinates
  - (a.k.a. Window Coordinates)

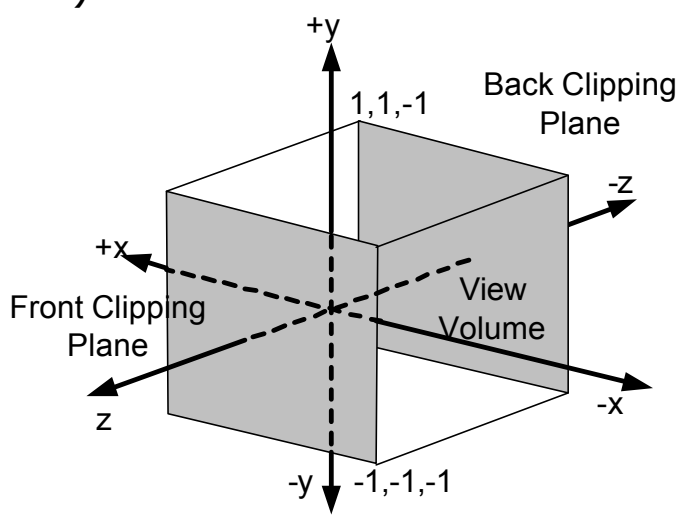


Idea: Canonical View Volume

- Define Viewing Volume via Canonical View Volumes
- Plus: Easier Clipping
- Minus: Another Transformation

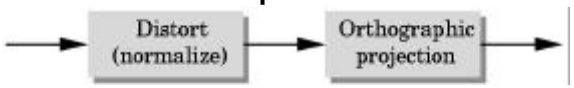


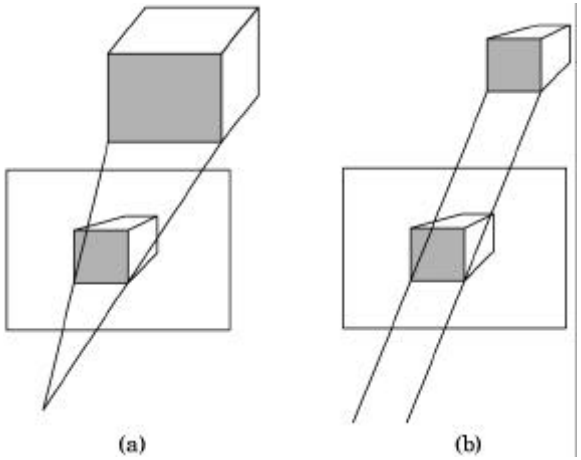
OpenGL's canonical view volume
(other APIs may be different):

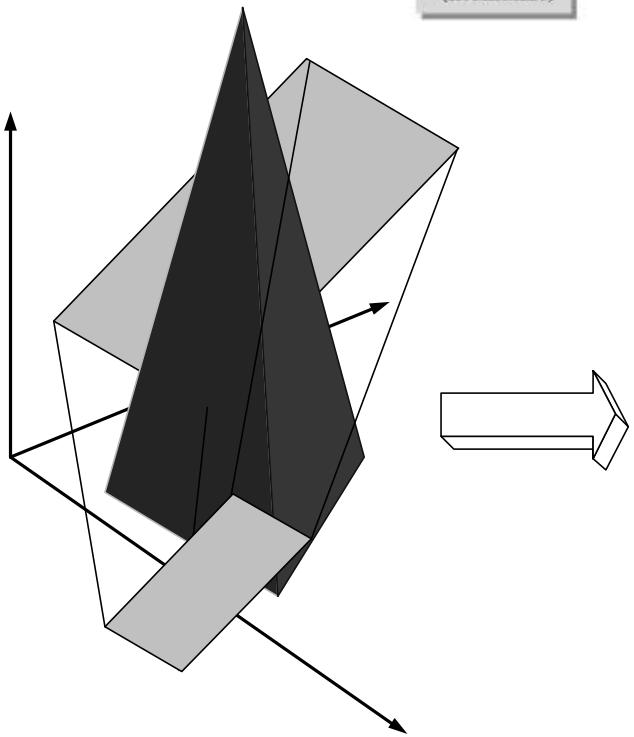


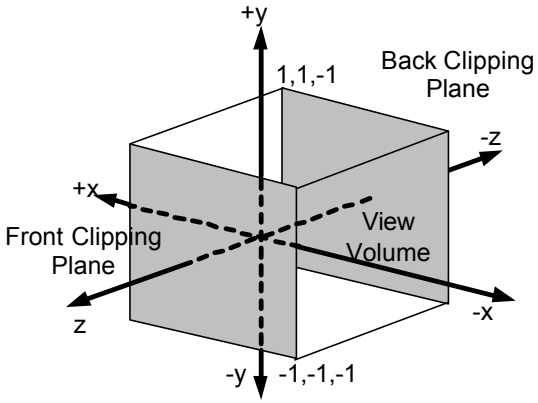
Projection Normalization

- Distort world until viewing volume in world fits into a parallel canonical view volume.
- Find a transformation that distorts the vertices in a way that we can use a simple canonical projection.



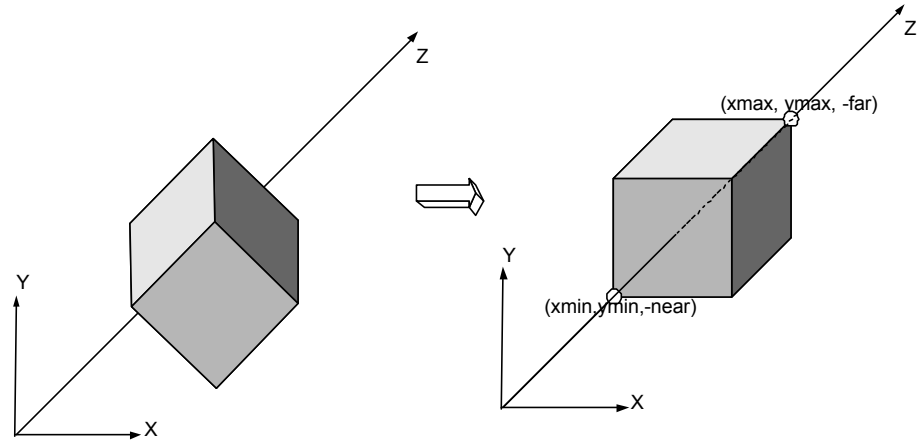




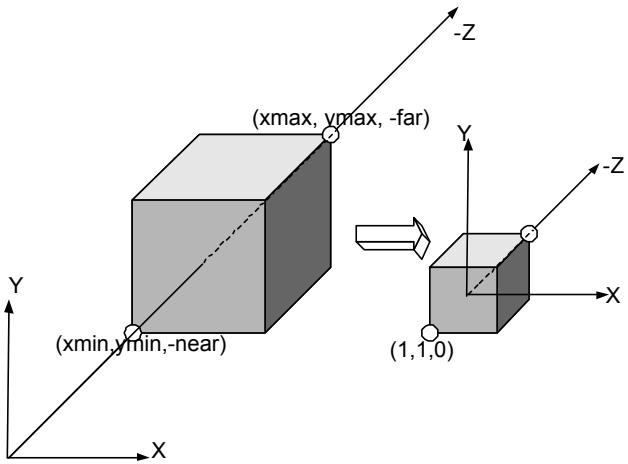


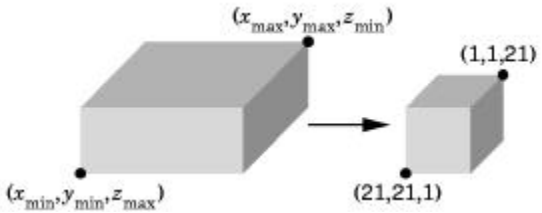
Camera Transformation and Projection Normalization for Orthogonal Views

- Camera Transformation in Orthogonal Views (M_{cam})
  1. Convert World to Camera Coordinates
     - Camera at origin, looking in the –z direction
     - Display plane center along the z axis
  2. Combinations of translate, scale, and rotate transformations
     - Can be accomplished through camera location specification



- Projection Normalization for Orthographic Views (P_{ortho})
  1. Translate along the z axis until the front clipping plane is at the origin
  2. Scale in all three dimensions until the viewing volume is in canonical form





Projection Normalization for Orthogonal Parallel Projections

Mapping a orthogonal view volume to a canonical view volume requires two affine transformations:

$$T_{\left(\frac{-(x_{\max}+x_{\min})}{2},\frac{-(y_{\max}+y_{\min})}{2},\frac{-(z_{\max}+z_{\min})}{2}\right)} \qquad S_{\left(\frac{2}{(x_{\max}-x_{\min})},\frac{2}{(y_{\max}-y_{\min})},\frac{2}{(z_{\max}-z_{\min})}\right)}$$

these can be concatenated to

$$P = ST = \begin{bmatrix} \frac{2}{x_{\max}-x_{\min}} & 0 & 0 & -\frac{x_{\max}+x_{\min}}{x_{\max}-x_{\min}} \\ 0 & \frac{2}{y_{\max}-y_{\min}} & 0 & -\frac{y_{\max}+y_{\min}}{y_{\max}-y_{\min}} \\ 0 & 0 & \frac{2}{z_{\max}-z_{\min}} & -\frac{z_{\max}+z_{\min}}{z_{\max}-z_{\min}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The camera is pointing in the negative z direction. All projectors are from infinity towards the origin. Hence P can be written as:

$$P = \begin{bmatrix} \frac{2}{x_{\max}-x_{\min}} & 0 & 0 & -\frac{x_{\max}+x_{\min}}{x_{\max}-x_{\min}} \\ 0 & \frac{2}{y_{\max}-y_{\min}} & 0 & -\frac{y_{\max}+y_{\min}}{y_{\max}-y_{\min}} \\ 0 & 0 & \frac{-2}{far-near} & \frac{far+near}{far-near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

OpenGL note:
OpenGL offers just an interface for the orthogonal case using
`glOrtho(xmin, xmax, ymax, ymin, near, far)`

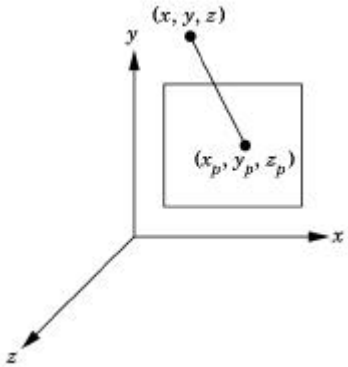
Projection Normalization for Orthogonal Parallel Projections

Finally, the resulting matrix has to be post multiplied by a simple orthogonal parallel projection

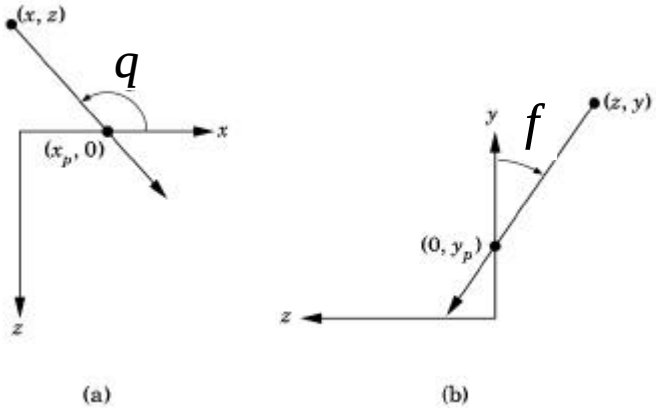
$$P = M_{orth}ST = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{x_{\max} - x_{\min}} & 0 & 0 & -\frac{x_{\max} + x_{\min}}{x_{\max} - x_{\min}} \\ 0 & \frac{2}{y_{\max} - y_{\min}} & 0 & -\frac{y_{\max} + y_{\min}}{y_{\max} - y_{\min}} \\ 0 & 0 & \frac{-2}{far - near} & \frac{far + near}{far - near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} \frac{2}{x_{\max} - x_{\min}} & 0 & 0 & -\frac{x_{\max} + x_{\min}}{x_{\max} - x_{\min}} \\ 0 & \frac{2}{y_{\max} - y_{\min}} & 0 & -\frac{y_{\max} + y_{\min}}{y_{\max} - y_{\min}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection Normalization for Oblique Parallel Projections



- Orthogonal parallel projection can be seen as just a special case of an oblique parallel projection.
- An oblique projection can be characterized by the angle of the projectors with the VP.



- Top and side views (see left) of a projector and the VP z=0.
- (q, f) characterize the degree of obliqueness.

Considering the top view (a), x_p can be found by

$$\tan q = \frac{z}{x - x_p} \Leftrightarrow x_p = x - z\cot q$$

and likewise following (b): $y_p = z - z\cot f$

For VP z=0 this results to P:

$$P = \begin{bmatrix} 1 & 0 & -\cot q & 0 \\ 0 & 1 & -\cot f & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After extracting the orthogonal projection we derive an additional shear: $P = M_{orth} H(q, f) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\cot q & 0 \\ 0 & 1 & -\cot f & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Projection Normalization for Oblique Parallel Projections

$$P = M_{orth} H(q,f) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\cot q & 0 \\ 0 & 1 & -\cot f & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

P is not in canonical form! It is a simple shear followed by an orthographic projection.

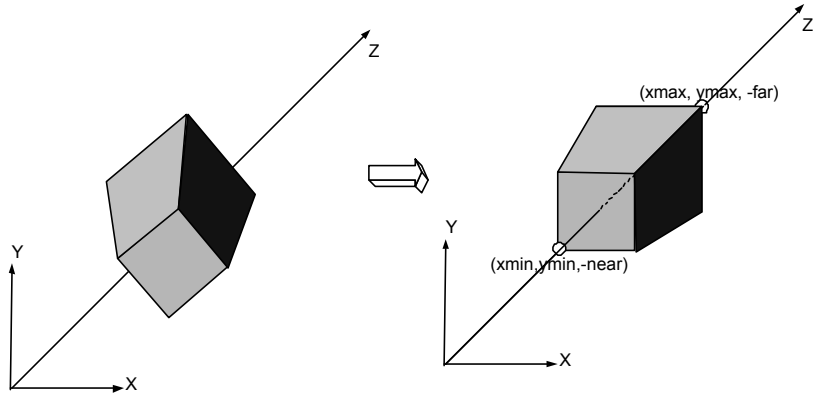
The same translation and scaling used for the orthographic case has to be inserted between the shear and the projection:

$$P = M_{orth} STH(q,f) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{x_{\max} - x_{\min}} & 0 & 0 & -\frac{x_{\max} + x_{\min}}{x_{\max} - x_{\min}} \\ 0 & \frac{2}{y_{\max} - y_{\min}} & 0 & -\frac{y_{\max} + y_{\min}}{y_{\max} - y_{\min}} \\ 0 & 0 & \frac{-2}{far - near} & \frac{far + near}{far - near} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\cot q & 0 \\ 0 & 1 & -\cot f & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

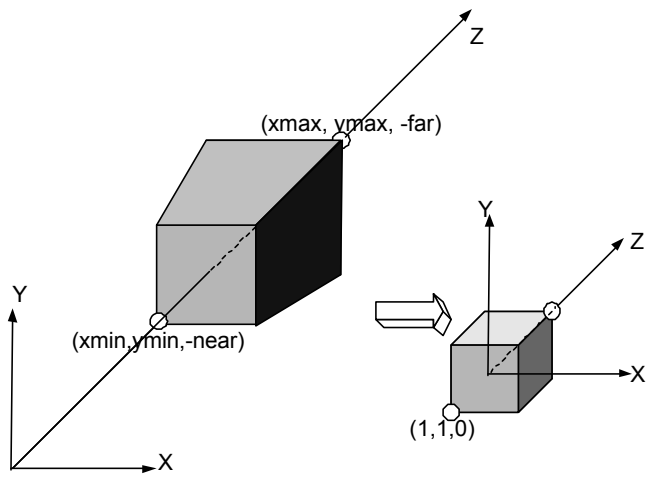
$$\Rightarrow P = \begin{bmatrix} \frac{2}{x_{\max} - x_{\min}} & 0 & \frac{-2\cot q}{x_{\max} - x_{\min}} & -\frac{x_{\max} + x_{\min}}{x_{\max} - x_{\min}} \\ 0 & \frac{2}{y_{\max} - y_{\min}} & \frac{-2\cot f}{y_{\max} - y_{\min}} & -\frac{y_{\max} + y_{\min}}{y_{\max} - y_{\min}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Camera Transformation (M_{cam}) and Projection Normalization (P_{pers}) for Perspective Views

- Camera Transformation for Perspective Views (M_{cam})
  1. Convert World to Camera Coordinates
     - Camera (COP) at origin, looking in the –z direction
     - Display plane center along the z axis
  2. Combinations of translate, scale, and rotate transformations
     - Can be accomplished through camera location specification



- Projection Normalization for Perspective Views (P_{pers})
  1. Convert viewing box to right frustum (on axis)
     - This is because many APIs including OpenGL allow non-right viewing volumes
  2. Scale the right frustum into canonical form
  3. Convert viewing box (right frustum) to a right parallelpiped
     - "Shrinking" objects that are further away



Projection Normalization for Perspective Projections

- Again, find a transformation that distorts the vertices in a way that we can use a simple canonical projection: **perspective-normalization transformation**

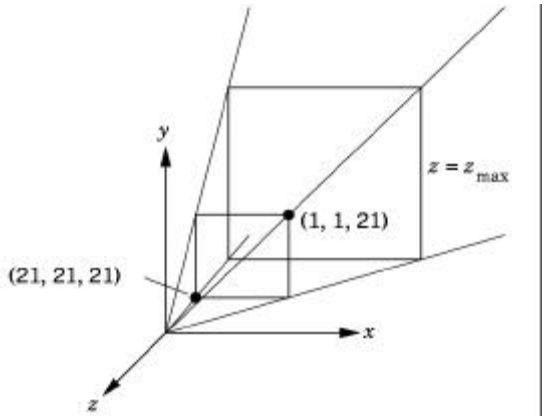
Given 1) a simple perspective projection with VP z=-1 and COP at origin:

$$M_{persp} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Given 2) a perspective view volume with the angle of view being 90° => frustum sides intersect VP at 45° angle. View volume is a semi infinite view pyramid with:

x = +/- z, y = +/- z

View volume is finite by specifying the near plane $z = z_{max}$ and the far plane $z = z_{min}$ with $z_{max} > z_{min}$



Projection Normalization for Perspective Projections

Let N be a nonsingular matrix similar to M with: $a \neq 0, b \neq 0$

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Applying N to a homogeneous-coordinate point:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ az+b \\ -z \end{bmatrix} \xrightarrow{\text{perspective division}} \begin{bmatrix} -\frac{x}{z} \\ -\frac{y}{z} \\ -\left(a+\frac{b}{z}\right) \\ 1 \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \\ z'' \\ 1 \end{bmatrix}$$

Applying an orthographic projection along the z-axis to N:

$$M_{orth} N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Applying the result to an arbitrary point p′:

$$M_{orth} N p' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ -z \end{bmatrix} \xrightarrow{\text{perspective division}} \begin{bmatrix} -\frac{x}{z} \\ -\frac{y}{z} \\ 0 \\ 1 \end{bmatrix}$$

If we apply N followed by an orthogonal projection to a point we achieve the same result for x and y as applying a perspective projection to the same point!

Projection Normalization for Perspective Projections

Nonsingular matrix N transforms the original viewing volume into a new volume.
Now choosing a, b such that the new volume is the canonical view (clipping) volume.
Given the sides

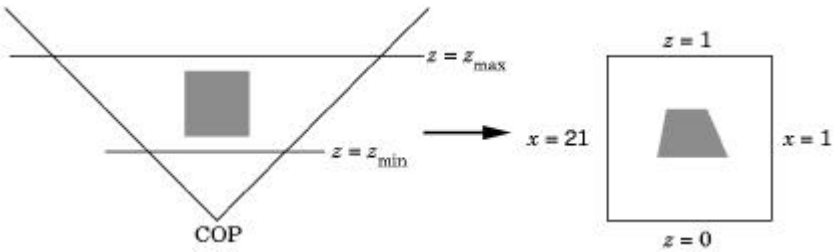
x = +/- z transformed by x″ results to x″ = +/- 1 and
y = +/- z transformed by y″ results to y″ = +/- 1 and

The front of the view volume $z=z_{max}$ is transformed to: $z'' = -\left(a + \frac{b}{z_{max}}\right)$

The back of the view volume $z=z_{min}$ is transformed to: $z'' = -\left(a + \frac{b}{z_{min}}\right)$

Now we choose: $a = \frac{z_{max} + z_{min}}{z_{max} - z_{min}}$ $b = -\frac{2 z_{max} z_{min}}{z_{max} - z_{min}}$

Then the plane z=z is mapped to the plane z″=-1 and the plane z=z is mapped to the plane z″=1, hence we achieve the canonical volume:



N transforms the viewing volume to a right parallelepiped, a following orthographic projection is the same as a perspective transformation. N is called the **perspective normalization matrix**.

Projection Normalization for Perspective Projections

$z'' = -\left(a + \frac{b}{z}\right)$ is nonlinear but preserves depth-ordering, hence $z_1 > z_2 \Rightarrow z''_1 > z''_2$

Notes:

- Hidden surface removal works in the normalized volume.
- Nonlinearity can cause numerical problems due to limited resolution in the depth buffer.
- Only one viewing pipeline is required by carefully choosing a projection matrix to insert into the pipeline.

- **Perspective-Normalization Matrix (N_{per})** converts frustum view volume into canonical orthogonal view volume:

$$N_{per} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{far + near}{far - near} & -\frac{2\, far \cdot near}{far - near} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Projection Normalization for non-right Perspective Projections

- The right (symmetric) perspective projection is a special case for an arbitrary perspective projection like the orthographic projection was for the parallel oblique case.
- An arbitrary perspective projection is required, e.g., for driving several large-screen projection-based VR display types which
  1. use head tracking and
  2. fix the VPs w.r.t. the moving COP

  and which hence require dynamic frustum calculation (responsive workbenches, Holoscreens, CAVEs,…)

  - This type of projection is a.k.a. **off-axis projection**!

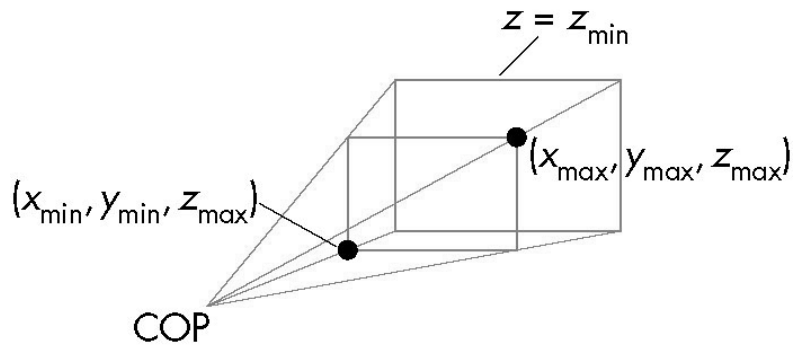
Do derive the projection matrix for off-axis set-ups we follow the same path as we did for the parallel projection case:

- Insertion of a shear transformation into the projection pipeline.

Projection Normalization for non-right Perspective Projections

Find H which satisfies:

$$H\begin{bmatrix} \frac{(x_{\min}+x_{\max})}{2} \\ \frac{(y_{\min}+y_{\max})}{2} \\ z_{\min} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z_{\min} \\ 1 \end{bmatrix}$$



$$H(q,f) = H\left(\cot^{-1}\left(\frac{x_{\min}+x_{\max}}{2z_{\max}}\right), \cot^{-1}\left(\frac{y_{\max}+y_{\min}}{2z_{\max}}\right)\right)$$

The resulting frustum is described by the planes:

$$x = \pm\frac{(x_{\max}-x_{\min})}{2z_{\max}}, y = \pm\frac{(y_{\max}-y_{\min})}{2z_{\max}}, z = z_{\min}, z = z_{\max}$$

Projection Normalization for non-right Perspective Projections

Now scale the sides to achieve $x = \pm z, y = \pm z$
without changing near/far planes .

$$S = S\left(\frac{2z_{\max}}{x_{\max} - x_{\min}}, \frac{2z_{\max}}{y_{\max} - y_{\min}}, 1\right)$$

S is without reference to z since it is uniquely determined by its results on four points, here the intersection points of the near plane and the sides. Now

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

gets the far plane to -1 and the near plane to 1 with the already chosen

$$a = \frac{z_{\max} + z_{\min}}{z_{\max} - z_{\min}} \quad b = -\frac{2z_{\max} z_{\min}}{z_{\max} - z_{\min}}$$

Projection Normalization for Perspective Views

- This results to the projection matrix:

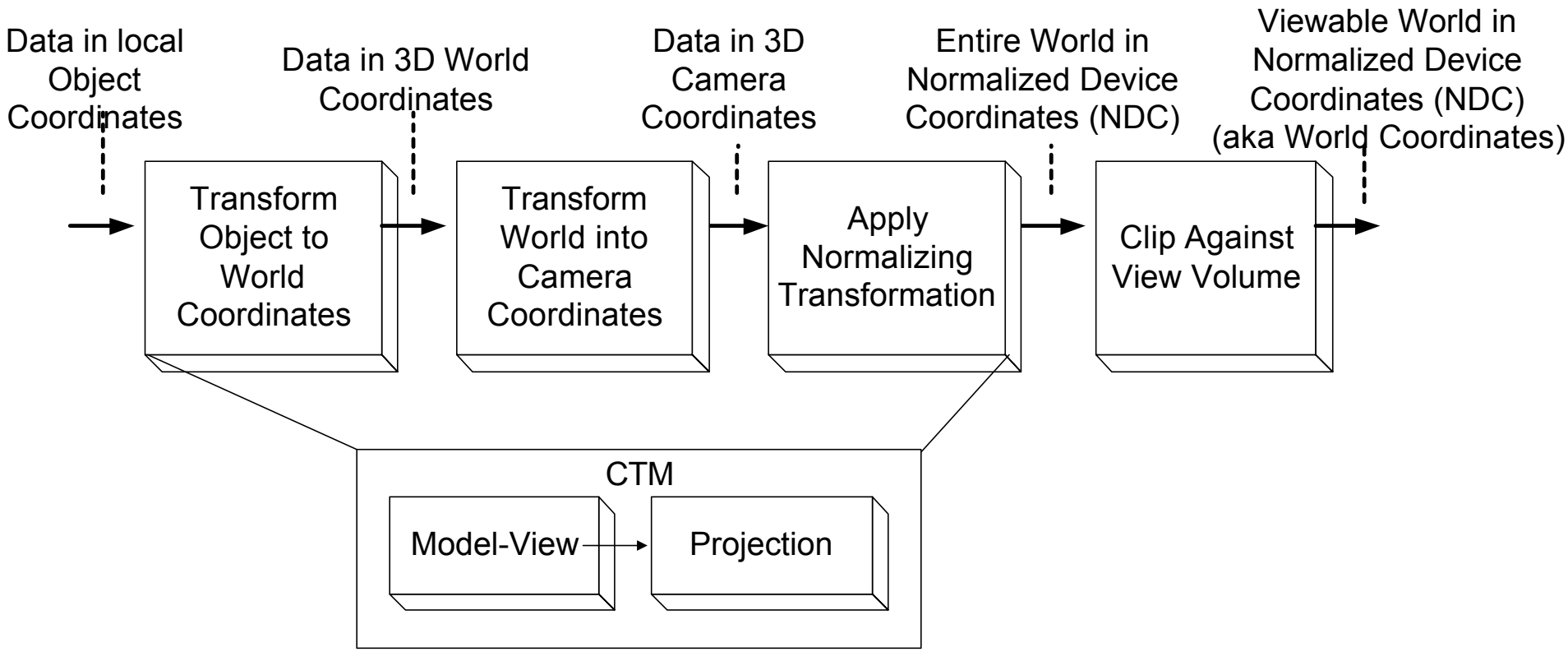
$$P_{pers} = N_{per} SH = \begin{bmatrix} \frac{2(-near)}{x_{\max} - x_{\min}} & 0 & \frac{x_{\max} + x_{\min}}{x_{\max} - x_{\min}} & 0 \\ 0 & \frac{2(-near)}{y_{\max} - y_{\min}} & \frac{y_{\max} + y_{\min}}{y_{\max} - y_{\min}} & 0 \\ 0 & 0 & -\frac{far + near}{far - near} & -\frac{2\, far \cdot near}{far - near} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Where H converts a non-right frustum to a right frustum
- Where S scales the frustum into a canonical perspective view volume
- Where N is the Perspective-Normalization Matrix

Project onto Projection Plane

- Since normalization changed all projections into an orthogonal projection:
  - Just ignore the z value!
  - In effect, a non-event!
- In reality, we retain the z-value for hidden-surface removal and shading effects.
- Viewable world now in Normalized Device Coordinates (NDC) or Window Coordinates

3D Viewing Summary



Matrix Operations

- Specify Current Matrix Stack

  `glMatrixMode(` *`GL_MODELVIEW`* `or` *`GL_PROJECTION`* `)`

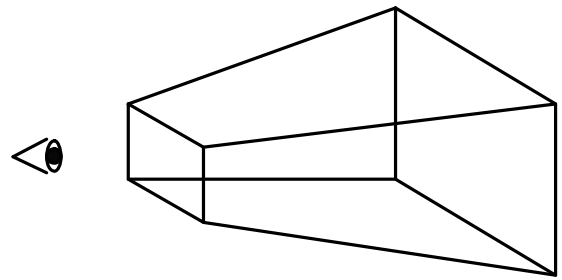
- Other Matrix or Stack Operations

  `glLoadIdentity()`
  `glPushMatrix()`
  `glPopMatrix()`

- Viewport
  - usually same as window size
  - viewport aspect ratio should be same as projection transformation or resulting image may be distorted

  `glViewport(` *`x, y, width, height`* `)`

Projection Transformation



- Shape of viewing frustum
- Perspective projection

  gluPerspective(***fovy, aspect, zNear, zFar*** **)**

  glFrustum(***left, right, bottom, top, zNear, zFar*** **)**
- Orthographic parallel projection

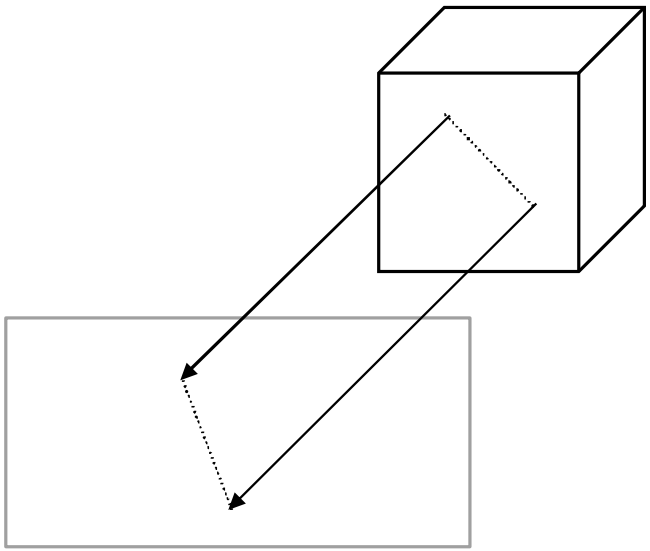
  glOrtho(***left, right, bottom, top, zNear, zFar*** **)**

  gluOrtho2D(***left, right, bottom, top*** **)**
  - calls **glOrtho** with z values near zero

Applying Projection Transformations

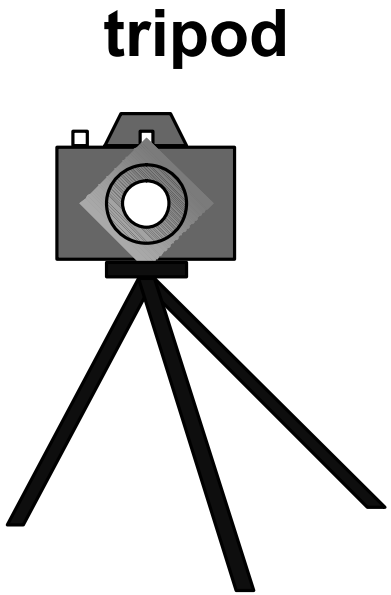
- Typical use (orthographic projection)

```
glMatrixMode( GL_PROJECTION );
glLoadIdentity();
glOrtho( left, right, bottom, top, zNear, zFar );
```

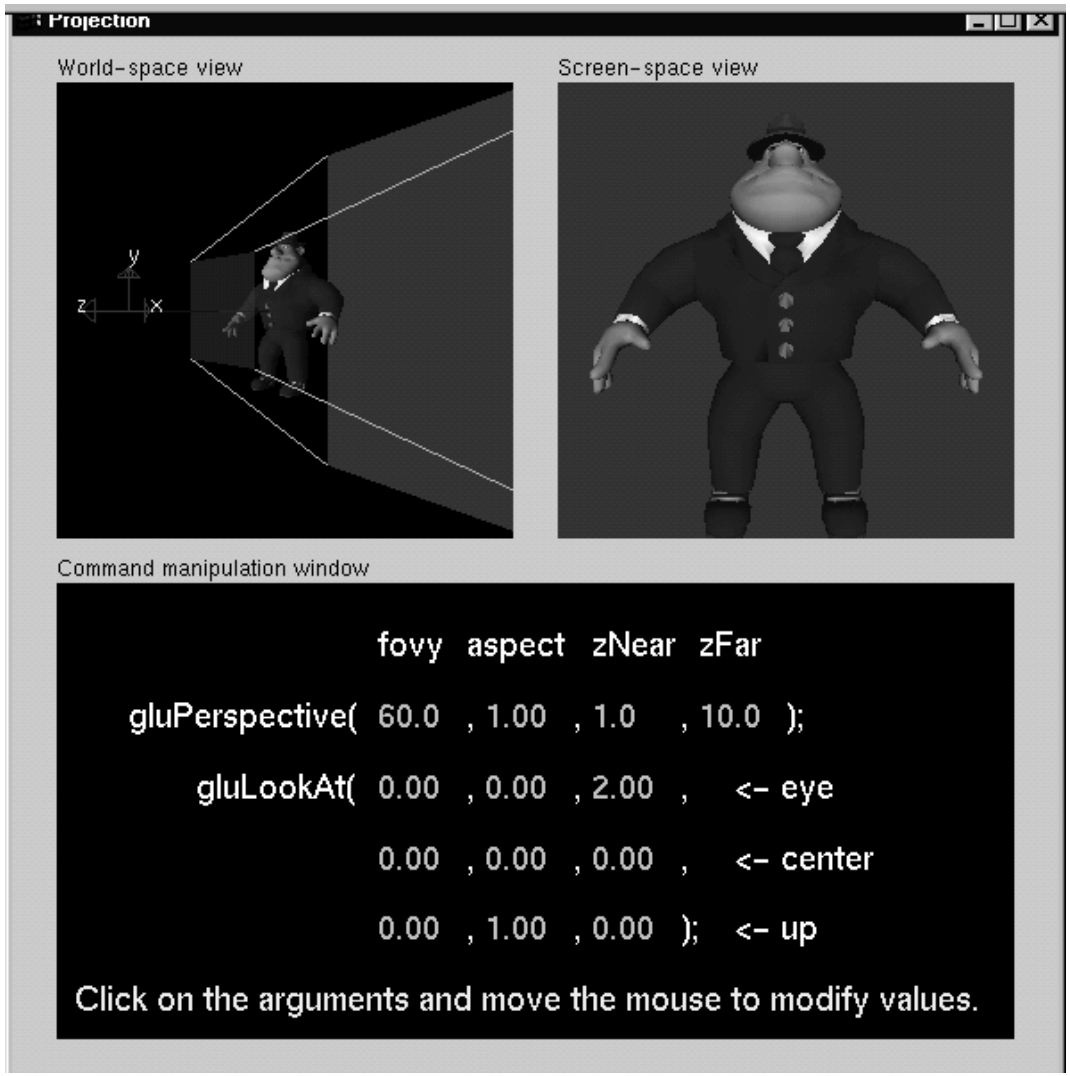


Viewing Transformations

- Position the camera/eye in the scene
  - place the tripod down; aim camera
- To "fly through" a scene
  - change viewing transformation and redraw scene
- `gluLookAt(` eye_x, eye_y, eye_z, aim_x, aim_y, aim_z, up_x, up_y, up_z `)`
  - up vector determines unique orientation
  - careful of degenerate positions



Projection Tutorial



Modeling Transformations

- Move object

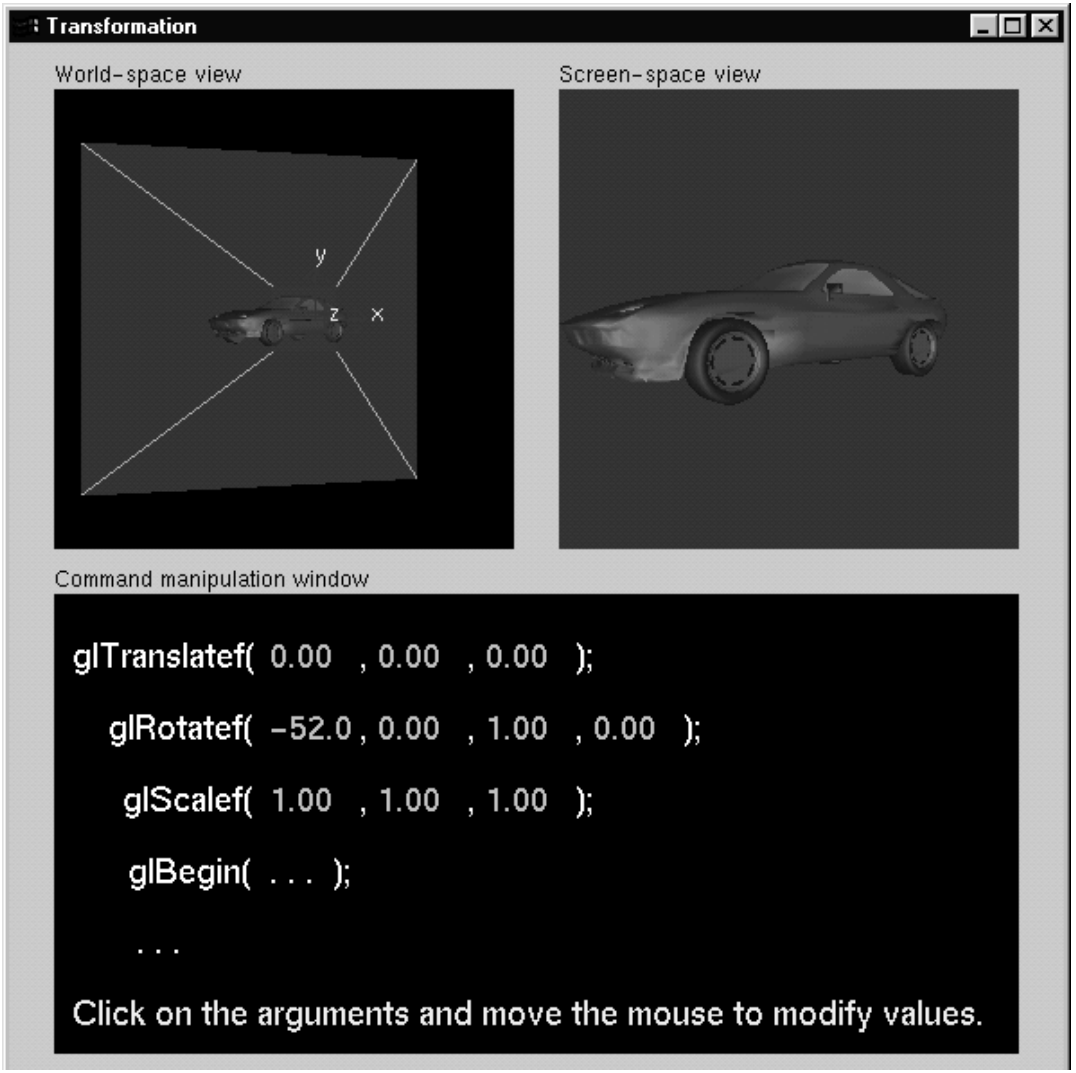
  `glTranslate{fd}(` ***`x, y, z`*** **`)`**
- Rotate object around arbitrary axis (x y z)

  `glRotate{fd}(` ***`angle, x, y, z`*** **`)`**

  – angle is in degrees
- Dilate (stretch or shrink) or mirror object

  `glScale{fd}(` ***`x, y, z`*** **`)`**

Transformation Tutorial

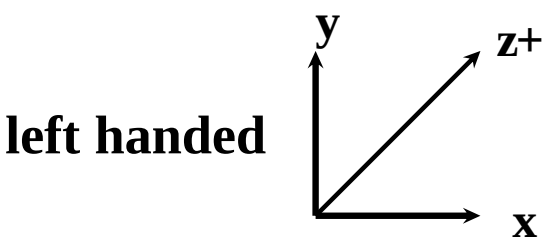


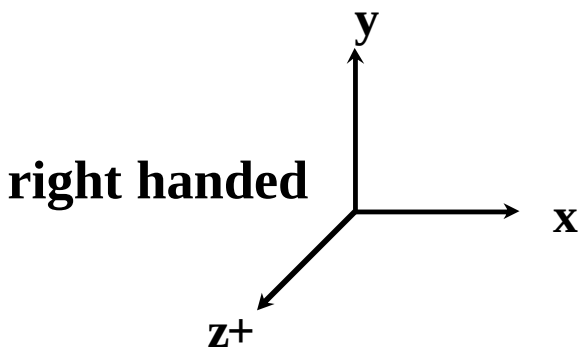
Connection: Viewing and Modeling

- Moving camera is equivalent to moving every object in the world towards a stationary camera
- Viewing transformations are equivalent to several modeling transformations
  - **`gluLookAt()`** has its own command
  - can make your own polar view or pilot view

Projection is left handed

- Projection transformations (**`gluPerspective, glOrtho`**) are left handed
  - think of ***`zNear`*** and ***`zFar`*** as distance from view point
- Everything else is right handed, including the vertexes to be rendered





Common Transformation Usage

- 3 examples of **`resize()`** routine
  - restate projection & viewing transformations
- Usually called when window resized
- Registered as callback for **`glutReshapeFunc()`**

resize(): Perspective & LookAt

```
void resize( int w, int h )
{
   glViewport( 0, 0, (GLsizei) w,
     (GLsizei) h );
   glMatrixMode( GL_PROJECTION );
   glLoadIdentity();
   gluPerspective( 65.0, (GLfloat) w / h,
                   1.0, 100.0 );
   glMatrixMode( GL_MODELVIEW );
   glLoadIdentity();
   gluLookAt( 0.0, 0.0, 5.0,
              0.0, 0.0, 0.0,
              0.0, 1.0, 0.0 );
}
```

`resize()`: Perspective & Translate

- Same effect as previous LookAt

```
void resize( int w, int h )
{
   glViewport( 0, 0, (GLsizei) w,
  (GLsizei) h );
   glMatrixMode( GL_PROJECTION );
   glLoadIdentity();
   gluPerspective( 65.0, (GLfloat) w/h,
                   1.0, 100.0 );
   glMatrixMode( GL_MODELVIEW );
   glLoadIdentity();
   glTranslatef( 0.0, 0.0, -5.0 );
}
```

`resize()`: Ortho (part 1)

```
void resize( int width, int height )
{
   GLdouble aspect = (GLdouble) width /
  height;
   GLdouble left = -2.5, right = 2.5;
   GLdouble bottom = -2.5, top = 2.5;
   glViewport( 0, 0, (GLsizei) w, (GLsizei) h
  );
   glMatrixMode( GL_PROJECTION );
   glLoadIdentity();
```

… continued …

resize(): Ortho (part 2)

```
    if ( aspect < 1.0 ) {
        left /= aspect;
        right /= aspect;
    } else {
        bottom *= aspect;
        top *= aspect;
    }
    glOrtho( left, right, bottom, top, near,
  far );
    glMatrixMode( GL_MODELVIEW );
    glLoadIdentity();
}
```

Compositing Modeling Transformations

- Problem 1: hierarchical objects
  - one position depends upon a previous position
  - robot arm or hand; sub-assemblies
- Solution 1: moving local coordinate system
  - modeling transformations move coordinate system
  - post-multiply column-major matrices
  - OpenGL post-multiplies matrices

Compositing Modeling Transformations

- Problem 2: objects move relative to absolute world origin
  - my object rotates around the wrong origin
    - make it spin around its center or something else
- Solution 2: fixed coordinate system
  - modeling transformations move objects around fixed coordinate system
  - pre-multiply column-major matrices
  - OpenGL post-multiplies matrices
  - must <u>reverse order of operations</u> to achieve desired effect

Additional Clipping Planes

- At least 6 more clipping planes available
- Good for cross-sections
- Modelview matrix moves clipping plane
- $Ax + By + Cz + D < 0$ clipped
- **`glEnable( GL_CLIP_PLANEi )`**
- **`glClipPlane( GL_CLIP_PLANEi, GLdouble* coeff )`**

Reversing Coordinate Projection

- Screen space back to world space

```
glGetIntegerv( GL_VIEWPORT, GLint viewport[4] )
glGetDoublev( GL_MODELVIEW_MATRIX, GLdouble mvmatrix[16] )
glGetDoublev( GL_PROJECTION_MATRIX,
                GLdouble projmatrix[16] )
gluUnProject( GLdouble winx, winy, winz,
                mvmatrix[16], projmatrix[16],
                GLint viewport[4],
                GLdouble *objx, *objy, *objz )
```

- `gluProject` goes from world to screen space