Universität Bielefeld

Realtime 3D Computer Graphics Virtual Reality

Basic mathematics

Algebraic structures - spaces
Matrix algebra
Vectors, points and frames of reference
Point transformation

Mathematics involved

The basic math for computer graphics deals with

- scalars: real numbers
- vectors: direction and magnitude
- matrices: 2 dimensional arrays of numbers
- points: positions in space
  (e.g., x, y, z measured w.r.t. a coordinate frame).

Three views on related concepts

1. The mathematical view
   - Scalars, points, vectors as member of mathematical sets.
   - Algebraic structures: Variety of abstract spaces and axioms for representing and manipulating these sets.
   - (linear) vector space, affine space, Euclidean space.
2. The geometric view
   - Mapping between the mathematical model and our perceived concept of space.
   - includes points as locations in space.
   - has referential properties (deixis)
3. Computer science view
   - See concepts as abstract data types (ADTs), a set of operations on data.
   - Use of geometric ADTs for points, vectors,…

Algebraic structures

- Def.: A **vector space** *V* is a set of vectors with the following attributes:

$$\forall \vec{u}, \vec{v} \in V : \vec{u} + \vec{v} \in V$$

$$\forall \vec{u} \in V, r \in \Re : r\vec{u} \in V$$

- A basis is a minimal set of vectors which span a vector space.
- (Note: No definition of distance, size, angles, or points)
- Def.: An **affine space** is a set of vectors *V* and a set of Points *P* for which:
  - *V* is a vector space and

$$\forall \vec{v} \in V, p \in P : p + \vec{v} \rightarrow q \in P$$

- An affine extension F (frame) of a basis is defined as:

$$F = (\vec{u}_i, i \in \Omega; o); \vec{u}_i \in V, basis; o \in P$$

Algebraic structures

- Def.: A **metric space** is a space with a metric d(p,q) for which:

$$d(p,q) \geq 0$$

$$d(p,q) = 0 \Leftrightarrow p = q$$

$$d(p,q) = d(q,p)$$

$$d(p,q) \leq d(p,r) + d(r,q)$$

- An **euclidian space** is a metric space with the following metric:

$$norm : |\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

$$angle : \alpha(\vec{u}, \vec{v}) = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}\right)$$

Algebraic structures

- Def.: A **cartesian space** is a euclidian space with an orthonormal basis:

$$orthogonal \ : \vec{u}_i \cdot \vec{v}_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \neq 0 & \text{sonst} \end{cases}$$

$$normal \ : \left|\vec{u}_i\right| = 1$$

- An **affine mapping** T maps between two affine spaces. Vectors map to vectors and points to points:
  - T is a linear transformation of vectors

$$T(p + \vec{u}) = T(p) + T(\vec{u})$$

$$\text{if} \quad \alpha + \beta = 1 : T(\alpha p + \beta q) = \alpha T(p) + \beta T(q)$$

otherwise relation must not hold!

Matrices

- Rectangular Array of Quantities (Row *(m)* by Column *(n)*).

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \end{bmatrix}$$

- Multiplication by a scalar:

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{bmatrix}$$

- Matrix addition (same number of rows/cols)
- Commutative: A+B = B+A
- Associative: A+(B+C) = (A+B)+C

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}$$

- Matrix multiplication
  Let A be a m x n matrix
  Let B be a n x q matrix
- C is a m x q matrix where: $C = AB \Rightarrow c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$
- Associative: A (B C) = (A B) C
- **Not Commutative: A B does not equal B A**

- Identity element for matrix multiplication. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- A I = I A

Matrices

- Matrix multiplication example:

$$\begin{bmatrix} 0 & -1 \\ 5 & 7 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0*1+-1*3 & 0*2+-1*4 \\ 5*1+7*3 & 5*2+7*4 \\ -2*1+8*3 & -2*2+8*4 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 26 & 38 \\ 22 & 28 \end{bmatrix} = \begin{bmatrix} -3 & 26 & 22 \\ -4 & 38 & 28 \end{bmatrix}^T$$

Matrix transpose

- Inverse matrix for matrix multiplication
  - Suppose we have a matrix multiplication of a square matrix *A* such that:
  - *Q = AP*
  - Do we have a square matrix *B* such that:
    - *P = BQ*
    - *P = BAP = IP = P => BA = I*
- If so, *B* is the inverse of *A* and *A* is *nonsingular.*
- The inverse of *A* is A^{-1}.

Affine mapping

- Affine = similar
- Set of mappings (continuous, bijective, invertible) $M_a : V^3 \rightarrow V^3$
- Invariants: straight lines, parallelism, fraction ratios
- Representation
  - Combination of a linear mapping L and a translation: $M_a(\vec{p}) = L\vec{p} + \vec{t}$
  - L is a 3x3 matrix
  - $\vec{t}$ is a translation vector
- Linear mappings are translation, rotation, scaling, and shear
- Remember: An **affine mapping** T maps between two affine spaces. Vectors map to vectors and points to points:
  - T is a linear transformation of vectors

    $T(p+\vec{u}) = T(p) + T(\vec{u})$

    if $\alpha + \beta = 1 : T(\alpha p + \beta q) = \alpha T(p) + \beta T(q)$

    otherwise relation must not hold!

Affine Transformation

- T is an affine transformation between two affine spaces A, B and

  $F_A = (\vec{v}_1, \vec{v}_2, o_A); F_B = (\vec{u}_1, \vec{u}_2, o_B);$

- p is a point from A with coordinates $(p_1, p_2, 1)$ $p = (p_1\vec{v}_1, p_2\vec{v}_2, 1o_A)$

- $$T(p) = T(p_1\vec{v}_1, p_2\vec{v}_2, 1o_A) = p_1 T(\vec{v}_1) + p_2 T(\vec{v}_2) + T(o_A)$$
  $$T(\vec{v}_i) = t_{i1}\vec{u}_1 + t_{i2}\vec{u}_2$$
  $$T(o_A) = t_{31}\vec{u}_1 + t_{32}\vec{u}_2 + o_B$$

Affine Transformation

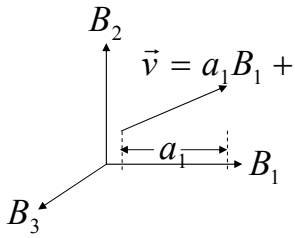
$$T(p) = T(p_1\vec{v}_1, p_2\vec{v}_2, 1o_A) = p_1 T(\vec{v}_1) + p_2 T(\vec{v}_2) + T(o_A)$$
$$T(p) = p_1(t_{11}\vec{u}_1 + t_{12}\vec{u}_2) + p_2(t_{21}\vec{u}_1 + t_{22}\vec{u}_2) + (t_{31}\vec{u}_1 + t_{32}\vec{u}_2 + o_B)$$
$$T(p) = \vec{u}_1(p_1 t_{11} + p_2 t_{21} + t_{31}) + \vec{u}_2(p_1 t_{12} + p_2 t_{22} + t_{32})o_B$$
$$T(p) = p'_1\vec{u}_1 + p'_2\vec{u}_2 + o_B$$

- Matrix representation

$$p = \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix}; \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$$ if basis is defined: $$: p = \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix}$$

$$T(p) = (\vec{v}_1, \vec{v}_2, o_B)\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix}$$

Coordinate Systems and frames

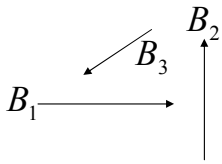


$$\vec{v} = a_1 B_1 + a_2 B_2 + a_3 B_3$$

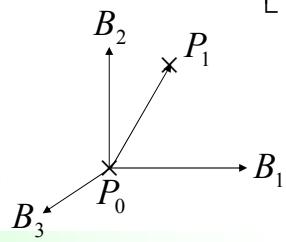
- In a 3 dimensional vector space every vector can be represented uniquely in terms of 3 linearly independent vectors.
- Leaving these 3 vectors fixed for reference, a vector can be represented just by its components.

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$



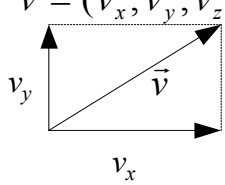
- Is there a problem with our definitions so far?
- No! We only need direction and magnitude for the vectors.
- With a point of reference we can uniquely address points as well.



$$P_1 = P_0 + b_1 B_1 + b_2 B_2 + b_3 B_3$$

Vectors in cartesian space

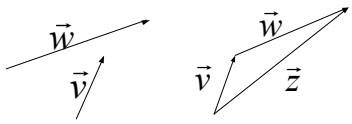
- A vector is a directed line segment
- Vectors have a direction and magnitude
  - **Vectors do not have a position!**
- One view is as a displacement between two points: $P_0 \xrightarrow{\vec{v}} P_1$
- All vectors can be broken down into their dimensional components vectors
  - 3D Vector: $\vec{v} = (v_x, v_y, v_z)$
  - 2D Vector:



- Magnitude is "length" of a vector: $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Vector Operations

- Multiply vector by scalar: $a\vec{v} = (av_x, av_y, av_z)$
- Add vectors: $\vec{z} = \vec{v} + \vec{w}$
  $= (v_x + w_x, v_y + w_y, v_z + w_z)$

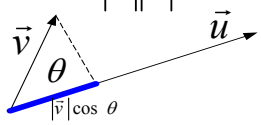


- Add vector to point: $P_2 = P_1 + \vec{v} \Rightarrow \begin{array}{l} P_{2x} = P_{1x} + v_x \\ P_{2y} = P_{1y} + v_y \\ P_{2z} = P_{1z} + v_z \end{array}$

Vector-Vector Multiplication

- Scalar product
  (aka) Dot Product
  - Product of two parallel components of the two vectors
  - Vector–vector multiplication producing a scalar

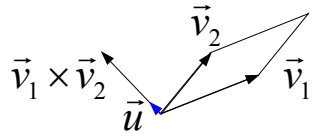
$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$



- For cartesian space
  (i.e. orthonormal base vectors):
  $\vec{v} \cdot \vec{w} = (v_x w_x + v_y w_y + v_z w_z)$

- Cross product
  - Produces normal vector perpendicular to plane formed
  - Magnitude equal to area of parallelogram formed
  - U is unit vector (magnitude 1) that is perpendicular to plane

$$\vec{v}_1 \times \vec{v}_2 = \vec{u}|\vec{v}_1||\vec{v}_2| \sin\theta$$



$$\vec{v} \times \vec{w} = (v_y w_z - v_z w_y, v_z w_x - v_x w_z, v_x w_y - v_y w_x)$$

Vectors, points, frames of reference

Homogeneous representation of vectors and points.

- Use of 3x1 matrices given a frame (v_1, v_2, v_3, P_0):

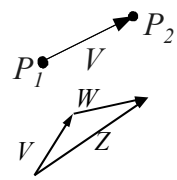
frame of reference point

$$P = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + P_0 \iff P = (\vec{v}_1, \vec{v}_2, \vec{v}_3, P_0)\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & 1\end{bmatrix}^T$$

$$\vec{w} = \delta_1\vec{v}_1 + \delta_1\vec{v}_2 + \delta_1\vec{v}_3 \iff \vec{w} = (\vec{v}_1, \vec{v}_2, \vec{v}_3, P_0)\begin{bmatrix}\delta_1 & \delta_2 & \delta_3 & 0\end{bmatrix}^T$$

vector

- The difference of two points is a vector:
  $(\alpha_1, \alpha_2, \alpha_3, 1) - (\beta_1, \beta_2, \beta_3, 1) = (\alpha_1 - \beta_1, \alpha_1 - \beta_2, \alpha_1 - \beta_3, 0)$
- The sum of a point and a vector is a point:
  $(\alpha_1, \alpha_2, \alpha_3, 1) + (\beta_1, \beta_2, \beta_3, 0) = (\alpha_1 + \beta_1, \alpha_1 + \beta_2, \alpha_1 + \beta_3, 1)$
- The sum of a vector and a vector is a vector:
  $(\alpha_1, \alpha_2, \alpha_3, 0) + (\beta_1, \beta_2, \beta_3, 0) = (\alpha_1 + \beta_1, \alpha_1 + \beta_2, \alpha_1 + \beta_3, 0)$



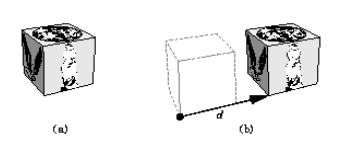
- Scaling a vector: $a * (\beta_1, \beta_2, \beta_3, 0) = (a\beta_1, a\beta_2, a\beta_3, 0)$
- Linear combination of vectors is valid

Transformations

- What are they?
  - Changing something to something else via rules.
  - Mathematics: mapping between values in a domain set and a range set (function/relation).
  - Geometric: translate, rotate, scale, shear,…
- Why are they important to graphics?
  - To move objects on screen / in space.
  - To map from model space to screen space.
  - To specify parent-child relationships.
  - …

➢ Remember the steps involved in 3D-CG pipeline.
➢ Each step uses transformations.
➢ Every transformation is equivalent to a change in coordinate systems (frames of reference).
➢ Geometric transformations can be modeled using linear algebra and matrix-matrix multiplication.

Translation

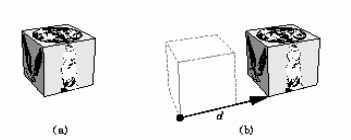


- Move object some distance along a displacement vector *d*.
- Rigid-Body Transformation (object does not change shape).
- Object is described by a set of points.
- Let p be one of those points.
- Let p' be this point after translation.

$$p = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, p' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}, d = \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \\ 0 \end{bmatrix} \qquad \begin{aligned} x' &= x + \alpha_x \\ y' &= y + \alpha_y \\ z' &= z + \alpha_z \end{aligned}$$

$$p' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x + \alpha_x \\ y + \alpha_Y \\ z + \alpha_z \\ 1 + 0 \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow T = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Translation



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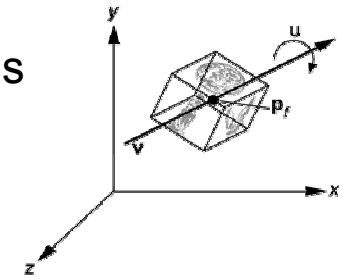
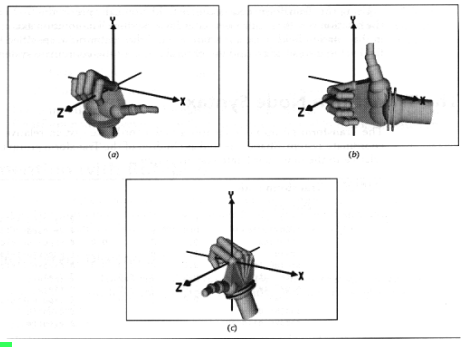
$$p' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x + \alpha_x \\ y + \alpha_Y \\ z + \alpha_z \\ 1 + 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_Y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow T = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_Y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$T(\alpha_x, \alpha_y, \alpha_z)$ can be reversed by: $T^{-1}(\alpha_x, \alpha_y, \alpha_z) = T(-\alpha_x, -\alpha_y, -\alpha_z)$

Rotation about the main axes

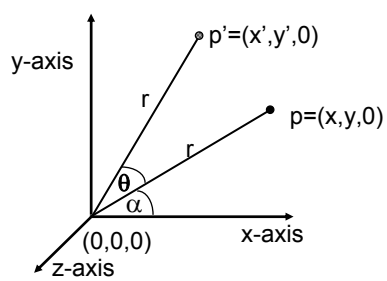


- Rotate the object about an axis
- Rigid-Body Transformation (object does not change shape)



- Use *right-hand rule* to determine rotation direction
  1. Close hand to loose fist with thumb pointing straight away
  2. Thumb direction is in the direction of the rotation axis.
  3. Finger direction (from palm to tips) describes positive rotation angles.

Rotation



Derivation of rotation around the Z axis:
Point p should be rotated by angle θ around the z-axis to result in p'.

$$p' = R_z p$$

$$\sin\alpha = {}^{y}\!/_{r}\,, \cos\alpha = {}^{x}\!/_{r}$$

$$\sin(\alpha+\theta) = {}^{y'}\!/_{r}\,, \cos(\alpha+\theta) = {}^{x'}\!/_{r}$$

$$\sin(\alpha+\theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$$

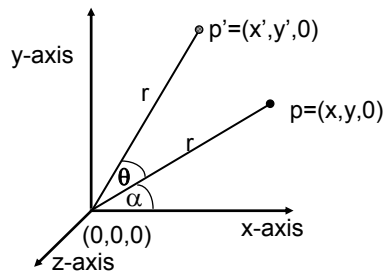
$$\cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta \quad \text{(double angle formulas)}$$

$$\Rightarrow {}^{x'}\!/_{r} = {}^{x}\!/_{r}\cos\theta - {}^{y}\!/_{r}\sin\theta \Leftrightarrow x' = x\cos\theta - y\sin\theta$$

$$\Rightarrow {}^{y'}\!/_{r} = {}^{y}\!/_{r}\cos\theta + {}^{x}\!/_{r}\sin\theta \Leftrightarrow y' = y\cos\theta + x\sin\theta \quad \text{and } z' = z$$

$$p' = R_z(\theta)p \Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} ??? & ??? & ??? & ??? \\ ??? & ??? & ??? & ??? \\ ??? & ??? & ??? & ??? \\ ??? & ??? & ??? & ??? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \\ 1 \end{bmatrix} \Rightarrow R_z(\theta) = \begin{bmatrix} ??? & ??? & ??? & ??? \\ ??? & ??? & ??? & ??? \\ ??? & ??? & ??? & ??? \\ ??? & ??? & ??? & ??? \end{bmatrix}$$

Rotation



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Point p should be rotated by angle θ around the z-axis to result in p'.

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$$\sin\alpha = y/r, \cos\alpha = x/r$$

$$\sin(\alpha+\theta) = y'/r, \cos(\alpha+\theta) = x'/r$$

$$\sin(\alpha+\theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$$

$$\cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta \quad \text{(double angle formulas)}$$

$$\Rightarrow x'/r = x/r\cos\theta - y/r\sin\theta \Leftrightarrow x' = x\cos\theta - y\sin\theta$$

$$\Rightarrow y'/r = y/r\cos\theta + x/r\sin\theta \Leftrightarrow y' = y\cos\theta + x\sin\theta \quad \text{and } z' = z$$

$$p' = R_z(\theta)p \Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \\ 1 \end{bmatrix} \Rightarrow R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation

- Rotation about the x axis:

$$p' = R_x p \quad where$$

$$R_x = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotation about the y axis:

$$p' = R_y p \quad where$$

$$R_y = R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

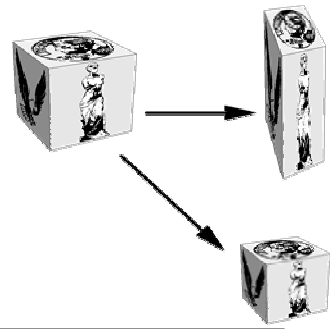
- Rotation about the z axis:

$$p' = R_z p \quad where$$

$$R_z = R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotation order is very important!
- Can be reversed by:
  $R^{-1}(\theta) = R(-\theta)$

Scaling

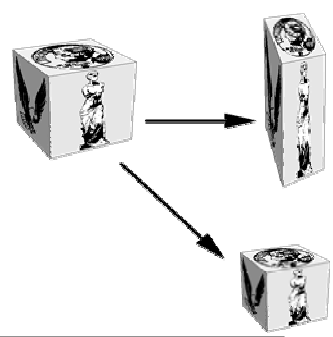


- Change the shape (size) by a given value.
- Non-Rigid Body Transformation.

$$x' = \beta_x x$$
$$y' = \beta_y y$$
$$z' = \beta_z z$$

$$p' = Sp \Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow S = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

Scaling



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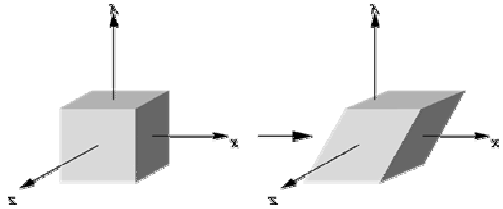
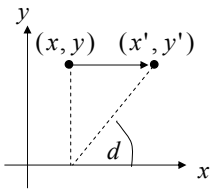
$S(\beta_x, \beta_y, \beta_z)$ can be reversed by: $S^{-1}(\beta_x, \beta_y, \beta_z) = S(1/\beta_x, 1/\beta_y, 1/\beta_z)$

Shear



- Change shape of object in an arbitrary direction.
- Non-rigid-body transformation.

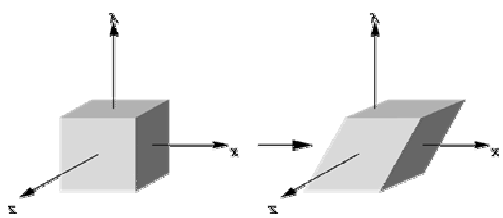


$$x' = x + y \cot d$$
$$y' = y$$
$$z' = z$$

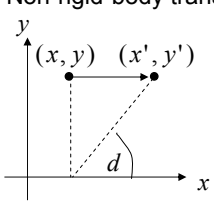
$p' = H_{xy}\, p$ *where* H_{xy} : x is target, y is source of shearing

$$p' = H(s)_{xy} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow H(s)_{xy} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}, s = \cot d$$

Shear



- Change shape of object in an arbitrary direction.
- Non-rigid-body transformation.



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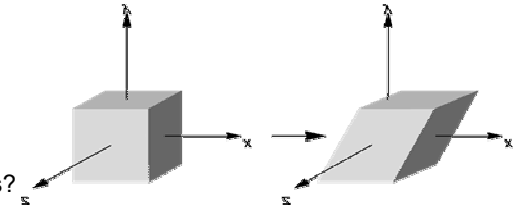
$$p' = H(s)_{xy} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \cot d & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow H(s)_{xy} = \begin{bmatrix} 1 & \cot d & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, s = \cot d$$

Shear



- How many shearing matrices exist if we share in planes orthogonal two main axis?
- There are six shearing matrices depending on the permutations of which axis is shear target and which is source.
- An alternative notation defines that two coordinates are sheared by the third:

$$p' = H_{xz}\,p \;\; where \quad H_{xz} = \begin{bmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$p' = H_{xy}\,p \;\; where \quad H_{xy} = \begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

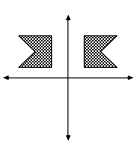
$$p' = H_{yz}\,p \;\; where \quad H_{yz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 0 \\ t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Can be reversed by:

$$H_x^{-1}(\theta) = H_x(-\theta)$$

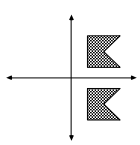
Reflections

2D Reflection about the y-axis



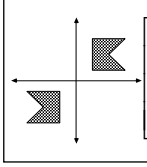
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2D Reflection about the x-axis



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2D Reflection about origin



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about the line y=x

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos 270 & -\sin 270 & 0 \\ \sin 270 & \cos 270 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

First rotate and then reflect: Concatenation of transformations (see next slides)