**Algorithmic Stochastics**  
**Exercise Sheet - Tutorial 9**  

**E17:** Let $P$ be a transition matrix on the finite set $S = \{1, 2, \ldots, N\}$. Show that if $\pi, \rho \in \mathbb{R}^N$ are two different stationary distributions of $P$. Show that for all $\alpha \in [0, 1]$, $v := \alpha \pi + (1 - \alpha)\rho$ is also a stationary distribution of $P$.

**E18:** A Markov chain $\{X_n\}_{n \geq 0}$ takes values in the state space $S = \{1, 2\}$ and has transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$$

a) Show by induction that the matrix of $n$-step transition probabilities is given by:

$$P^n = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix} + \left( -\frac{1}{2} \right)^n \begin{pmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{pmatrix}$$

b) Verify that the chain is irreducible, and find the stationary distribution $\pi = (\pi_1, \pi_2)$ for the chain.

c) Verify that for all $i, j \in \{1, 2\}$

$$\lim_{n \to \infty} p_{ij}^n = \pi_j.$$ 

**E19:** We have two red balls, two green balls, and two boxes labeled $A$ and $B$. Each box contains 2 balls. The state of the system is entirely described by the number of red balls in box $A$. The state of the system is changed by choosing a ball from each box at random and placing the ball from box $A$ into box $B$ and the ball from box $B$ into box $A$. Let $X_n$ be the state of the system after $n$ switches have been made.

a) Write down the state space.

b) Show that this process is a Markov chain and find the transition matrix for this Markov chain.

c) Verify whether it is aperiodic and irreducible or not.

d) Answer the same question if we change to three red balls, three green balls and three balls in each box.

**E20:** Suppose that the sequence $X_0, X_1, X_2, \ldots$ of random variables with values in $\mathbb{Z}$ has the Markov property.

a) Show that the sequence $X_0^3, X_1^3, X_2^3, \ldots$ also has the Markov property.

b) Give an example where the sequence $X_0^2, X_1^2, X_2^2, \ldots$ does not have the Markov property.  

*Hint:* If $x^3 = y$ then $x = y^{1/3}$. For example, $x^3 = -8$ means that $x = -2$. But if $x^2 = y$, then $x$ equals $\sqrt{y}$ or $-\sqrt{y}$. For example, $x^2 = 100$ means that $x = 10$ or $x = -10$. 
