Mathematical Biology
Faculty of Technology, Biomathematics and Theoretical Bioinformatics

Exercises

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Contents

**Home exercise 1**

Exercise 1.1 IVP \( x = x(t) \cdot f(t), \ x(0) = x_0 \) ........................................ 1 – 1
Exercise 1.2 Check solution of a 2nd order ODE .................................................. 1 – 1
Exercise 1.3 Spreading of a disease ................................................................. 1 – 2
Exercise 1.4 Logistic ODE: Check solution of IVP .......................................... 1 – 2
Exercise 1.5 Solution \( x(t) = 2e^t - 1 \) given: find IVP ........................................ 1 – 2

**Home exercise 2**

Exercise 2.1 Calcute eigensystem ................................................................. 2 – 1
Exercise 2.2 Carrion eater-hyena-model ......................................................... 2 – 2

**Home exercise 3**

Exercise 3.1 Solution in the plane ................................................................. 3 – 1
Exercise 3.2 Generalised logistic ODE .......................................................... 3 – 2
Exercise 3.3 System of ODEs ........................................................................... 3 – 2

**Home exercise 4**

Exercise 4.1 SI-model ....................................................................................... 4 – 1
Exercise 4.2 SIR-model ..................................................................................... 4 – 2

**Home exercise 5**

Exercise 5.1 Blood-cell model ........................................................................... 5 – 1
Exercise 5.2 Diploid selection equation .......................................................... 5 – 2
Exercise 5.3 BRN SIR ........................................................................................ 5 – 2

**Home exercise 6**

Exercise 6.1 IVP \( \dot{y} = cy^2 \), determine solution and validate it ............... 6 – 1
Exercise 6.2 Vaccination ..................................................................................... 6 – 2
Exercise 6.3 Exponential transformation ......................................................... 6 – 2
Exercise 1.1  IVP \( \dot{x} = x(t) \cdot f(t), \ x(0) = x_0 \)

As a generalisation of the differential equation defined in the lecture, consider now the initial value problem
\[
\dot{x}(t) = x(t) \cdot f(t), \ x(0) = x_0
\]
(with a time dependent function \( f(t) \)). Its solution reads
\[
x(t) = x_0 e^{\int_0^t f(\tau) d\tau}.
\]

Subtask 1.1.1 Verification of the statement
Verify the statement.

Subtask 1.1.2 Derivation/separation of variables
Derive the solution in a constructive way
(by using separation of variables, which will be explained by the tutor).

[Hint: \( \frac{\dot{x}(t)}{x(t)} = \frac{d}{dt} \log x(t) \)]]

Exercise 1.2  Check solution of a 2nd order ODE

Let the function \( g : \mathbb{R} \to \mathbb{R} \) be twice differentiable with \( g'(x) \neq 0 \) for all \( x \in \mathbb{R} \). Furthermore, let the function \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = \cos(kg(x)) \), where \( k \in \mathbb{R} \). Show that
\[
f'' - f'\frac{g''}{g'} + (kg')^2 f = 0.
\]
Home exercise

Exercise 1.3  Spreading of a disease

We want to describe the spreading of an infectious disease, which is transmitted at rate \( \alpha \) if an infected individual meets a noninfected one, and from which infected individuals recover at rate \( \mu \). Let \( p \) be the proportion of infected individuals in a population; then \( 1 - p \) is the proportion of noninfected ones. Since infections require contact between infected and noninfected individuals, the increase of the proportion of infected individuals is proportional to both \( p \) and \( 1 - p \); the constant of proportionality is \( \alpha \). The loss of infected individuals is only proportional to \( p \) with constant of proportionality \( \mu \). Altogether, \( p \) changes at rate

\[
\dot{p} = \alpha p (1 - p) - \mu p
\]

Subtask 1.3.1  Phase line diagrams, 1 point

Draw the phase line diagrams for \( \alpha < \mu \) and \( \alpha > \mu \). What follows for the qualitative behavior (equilibria, stability)? Sketch selected solutions.

Subtask 1.3.2  Discussion state of health, 1 point

Discuss what the two cases mean for the state of ’health’ of the population and the spreading of the disease?

Exercise 1.4  Logistic ODE: Check solution of IVP, 1 point

Consider the logistic differential equation, this time in the form

\[
\dot{x} = \lambda x \frac{K - x}{K}
\]

Verify that the function

\[
x(t) = \frac{K x_0}{x_0 + (K - x_0) e^{-\lambda t}}
\]

is the solution of this differential equation with initial value \( x_0 \).

[Hint: Differentiate and have a sharp look at the resulting expression. Don’t expand in any case!]

Exercise 1.5  Solution \( x(t) = 2e^t - 1 \) given: find IVP, 1 point

Find the initial value problem that is solved by \( x(t) = 2e^t - 1 \)
Mathematical Biology 2

Submission of your solutions: 18.04.2019 (in the lecture)

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Summerterm 2019

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Presence exercise

Exercise 2.1  Calculate eigensystem

Calculate the eigenvalues and (right) eigenvectors of the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 - \alpha & \beta \\ \alpha & 1 - \beta \end{pmatrix}$$
**Home exercise**

**Exercise 2.2** Carrion eater-hyena-model

Consider the behaviour of two competing species, i.e. carrion eater and hyenas. The population size of the carrion eater at timepoint \( t \) is denoted by \( A(t) \), those of the hyenas by \( H(t) \). Both species compete more or less for the same ressource. The following equations may serve to describe the dynamic of the population sizes:

\[
\frac{dA}{dt} = A - (A^2 + \alpha AH) \\
\frac{dH}{dt} = H - (H^2 + \alpha HA)
\]

with the additional condition that \( 0 < \alpha \).

**Subtask 2.2.1 Equilibria, 1 point**

Calculate all equilibria.

**Subtask 2.2.2 Graphical analysis, 1 point**

Draw the nullisoclines as well as the vector field sketch in the case \( \alpha = 2 \). Sketch the trajectories in the case \( \alpha = 2 \) for an initial value \((A_0, H_0)\) with \( A_0 < H_0 \). Conclude the stability of the equilibria for this \( \alpha \) with the help of your sketch.

**Subtask 2.2.3 Analysis via Jacobian, 3 points**

Validate the stability of the equilibria for an arbitrary \( \alpha \neq 1 \) by using the Jacobian matrix. Which case distinction is necessary? What can you conclude for the long time development of both species from your results?
Mathematical Biology 3

Submission of your solutions: 26.04.2019 (in the lecture)

Mathematical Biology
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Presence exercise

Exercise 3.1  Solution in the plane

Consider the following solution of a differential equation in the plane:

Subtask 3.1.1  Coordinates as functions of $t$, 1 point

Draw the corresponding time courses $x(t)$, $y(t)$, as precisely as possible. The 11 time points are equidistant.

Subtask 3.1.2  Possible nullisoclines and ODE system, 3 points

Draw possible nullisoclines in the picture and set up an associated possible system of differential equations.
Home exercise

**Exercise 3.2** Generalised logistic ODE

Consider the following differential equation, which describes the size of a population:

\[ \dot{x} = -rx \left(1 - \frac{x}{T}\right) \left(1 - \frac{x}{K}\right) \]

with \(0 < T < K\).

**Subtask 3.2.1** Phase lines, equilibria, stability, 2 points

First draw the phase line diagram and use it to conclude the stability of the equilibria. Then verify the stability properties by analysing the derivative of the right-hand side at equilibrium.

**Subtask 3.2.2** Time course of solutions, long time behaviour, 1 point

Sketch the time course of the solution for \(0 < x_0 < T\), \(T < x_0 < K\), and \(x_0 > K\), and draw conclusions about the long-term behaviour of the size of the population. Interpret the meaning of the parameter \(T\).

**Exercise 3.3** System of ODEs

Consider the ODE system

\[ \begin{align*}
\dot{x} &= g(x, y) = 5 - x - xy + 2y \\
\dot{y} &= h(x, y) = xy - 3y.
\end{align*} \]

**Subtask 3.3.1** Equilibria, 1 point

Calculate the equilibria. (Hint: factorise \(h\) and insert its solutions (individually) into \(g\) (\(g\) cannot be factorised)).

**Subtask 3.3.2** Nullisoclines, 1 point

Solve \(g\) for \(y\) to obtain the \(x\) nullisocline as a function of \(x\). This function has a vertical and a horizontal asymptote; which ones? What kind of function is the \(x\) nullisocline?

Draw both nullisoclines as well as the equilibria.

**Subtask 3.3.3** Vector field sketch, 1 point

Determine the signs of \(\dot{x}\) and \(\dot{y}\) in the positive quadrant (i.e. for \(x, y > 0\)). (Hint: A case distinction is required.)

Sketch the corresponding vector field. Can you conclude the stability of the equilibrium in the positive quadrant?
Mathematical Biology 4

Submission of your solutions: 03.05.2019 (in the lecture)

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Presence exercise

Exercise 4.1 SI-model

Consider the following infection model:

\[
\begin{align*}
\dot{I} &= \alpha IS - \mu I \\
\dot{S} &= -\alpha IS + \rho S \left(1 - \frac{I + S}{K}\right)
\end{align*}
\]

Here \( I \) denotes the number of infected, \( S \) the number of susceptible individuals.

Subtask 4.1.1 Description of the model

Which situation ist described by the model? What meaning do the parameters \( \alpha, \mu, \rho, K \) have?

Subtask 4.1.2 Nullisoclines, equilibria, vector field, stability

Calculate and draw the nullisoclines and the equilibria and sketch the vector field in the positive quadrant. Can you infer the stability of the internal equilibrium (that is, the one with both components positive)?
Exercise 4.2  SIR-model

Let $S(t)$ be the number of individuals that can be infected with a disease (susceptibles), $I(t)$ be the number of those that are already infected (infecteds) and $R(t)$ be the number of those that were infected and are recovered now (recovered). $\beta$, $\nu$ and $\gamma$ are positive parameters. The interplay of the three groups may be described by a simple epidemiological model

$$
\frac{dS}{dt} = -\beta \frac{S}{N} I + \gamma R,
\frac{dI}{dt} = \beta \frac{S}{N} I - \nu I,
\frac{dR}{dt} = \nu I - \gamma R.
$$

Subtask 4.2.1  constant population size, 1 point

Show that the total population size,

$$N(t) := S(t) + I(t) + R(t),$$

is constant over time.

Subtask 4.2.2  Assumptions and reduction, 1 point

Interpret the equations in terms of the basic assumptions of the model; in particular, describe the meaning of the parameters. Then, reduce the model to a system of two coupled differential equations. For this purpose, use the additional condition in the form $R = N - I - S$.

Subtask 4.2.3  Equilibria, stability, 3 points

Calculate the equilibria of the reduced model. Use the Jacobian matrix to examine the equilibrium $(\bar{S}, \bar{I}) = (N, 0)$ with respect to stability. Under which condition is it attractive? Interpret your result.

Subtask 4.2.4  Enhancement to birth-death process, 1 point

The above model is unrealistic in various ways. Generalise the system of equations by including births and deaths of individuals. Use $\mu$ as a constant rate of birth and death per individual. Which assumption do you make?
Mathematical Biology 5

Submission of your solutions: 10.05.2019 (in the lecture)

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Presence exercise

Exercise 5.1  Blood-cell model

Most types of blood cells are formed from primitive bone marrow stem cells. Until today, the exact production process of blood cells has not yet been sufficiently understood. However, it is known that the production rate depends on the cell density $y(t)$. A model that has very well described the measured cell density is based on the equation

$$\dot{y} = \frac{b\theta^m y}{\theta^n + y^n} - cy = p(y) - cy,$$

where $b$, $\theta$, $c$, $n > 1$ are positive parameters with $b \neq c$ and $p(y) = \frac{b\theta^m y}{\theta^n + y^n}$ indicating the production rate of blood cells.

Subtask 5.1.1  Transformation of the differential equation

Show that, via the substitution $y = u\theta$, the equation above may be transformed into:

$$\dot{u} = \frac{bu}{1 + u^n} - cu.$$  \hspace{1cm} (1)

Subtask 5.1.2  Equilibria, stability

Find all equilibria and calculate their stability.
**Home exercise**

**Exercise 5.2** Diploid selection equation

Consider the following differential equation

\[ \dot{x} = \gamma x^2 (1 - x) - u x = g(x) \]

with parameters \( u, \gamma > 0 \).

(This is the so-called diploid selection equation for a recessive allele.)

Subtask 5.2.1 , 1 point

Determine the equilibria; distinguish the cases \( u < \frac{1}{4} \gamma \) and \( u > \frac{1}{4} \gamma \) (you need not consider the case \( u = \frac{1}{4} \gamma \)).

Subtask 5.2.2 , 1 point

Sketch the phase line diagram and determine the stability of the equilibria, individually for the two cases in 5.2.1. *(Hint: Since \( g \) is a polynomial, 5.2.1 already gives you the required information; no further calculation is needed.)*

Subtask 5.2.3 , 1 point

Represent the equilibria and their stabilities graphically as a function of \( u \).

**Exercise 5.3** BRN SIR

Subtask 5.3.1 , 1 point

Consider again the SIR model of Exercise 4.2.

Calculate its basic reproduction number, that is, the mean number of secondary cases induced by a single infected individual introduced into an otherwise susceptible population.

Subtask 5.3.2 , 1 point

Consider now the following modified version of the SIR model

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI + \gamma R \\
\frac{dI}{dt} &= \beta SI - \nu I \\
\frac{dR}{dt} &= \nu I - \gamma R,
\end{align*}
\]

again with \( N(t) := I(t) + S(t) + R(t) \equiv N \).

Calculate \( R_0 \) for this model. What is different, and why?
Mathematical Biology 6

Submission of your solutions: 17.05.2019 (in the lecture)

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Summerterm 2019

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Presence exercise

Exercise 6.1  IVP $\dot{y} = cy^2$, determine solution and validate it

Solve the initial value problem

$$\dot{y} = cy^2, \quad y(t_0) = y_0 > 0, \quad c > 0$$

via separation of variables. Does the solution exist for all $t > t_0$?
Home exercise

**Exercise 6.2**  Vaccination , 1 point

Consider some infection model with a given $R_0$. Consider now the case that at the beginning a share $v$ of the population is vaccinated. What is the value of the new reproduction number, $R_v$? How big must $v$ be to avoid an outbreak? Calculate the proportion of vaccinated people necessary to prevent the spread of the disease. Evaluate this proportion explicitly for the case of measles ($R_0 = 15$ without vaccination) and smallpox ($R_0 = 6$ without vaccination).

**Exercise 6.3**  Exponential transformation , 4 points

Consider the differential equation

$$\dot{y} = -sy(1-y) + u(1-y) - vy, \quad y \in [0,1].$$

Consider now the following quantities obtained from $y$ via

$$\begin{align*}
z_0(t) &:= (1 - y(t)) f(t) \\
z_1(t) &:= y(t) f(t)
\end{align*}$$

where $f(t) = e^{\int_0^t s(1-y(\tau))d\tau}$.

Find the system of differential equations that is satisfied by $z(t) = (z_0(t)z_1(t))$.

Interpret this system in terms of a population model with two types of individuals that reproduce and mutate.

Express $y(t)$ and $1 - y(t)$ as functions of $z(t)$. So what is the meaning of $y(t)$ and $1 - y(t)$ in terms of the population model?

Also give an interpretation of $f(t)$ in terms of the population model.