Mathematical Biology
Faculty of Technology, Biomathematics and Theoretical Bioinformatics

Exercises

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Summerterm 2019
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Mathematical Biology 1

**Ordinary differential equations**

Submission of your solutions: 12.04.2019 (in the lecture)

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**Presence exercise**

**Exercise 1.1** IVP \( \dot{x} = x(t) \cdot f(t), \ x(0) = x_0 \)

As a generalisation of the differential equation defined in the lecture, consider now the initial value problem

\[ \dot{x}(t) = x(t) \cdot f(t), \ x(0) = x_0 \]

(with a *time dependent* function \( f \)). Its solution reads

\[ x(t) = x_0 e^\int_0^t f(\tau) d\tau. \]

Subtask 1.1.1 Verification of the statement
Verify the statement.

Subtask 1.1.2 Derivation/separation of variables
Derive the solution in a constructive way
(by using separation of variables, which will be explained by the tutor).

\[ \text{[Hint: } \frac{\dot{x}(t)}{x(t)} = \frac{d}{dt} \log x(t) \text{] [!] \]

**Exercise 1.2** Check solution of a 2nd order ODE

Let the function \( g : \mathbb{R} \rightarrow \mathbb{R} \) be twice differentiable with \( g'(x) \neq 0 \) for all \( x \in \mathbb{R} \). Furthermore, let the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( f(x) = \cos(kg(x)) \), where \( k \in \mathbb{R} \). Show that

\[ f'' - f' \frac{g''}{g'} + (kg')^2 f = 0. \]
Home exercise

Exercise 1.3  Spreading of a disease

We want to describe the spreading of an infectious disease, which is transmitted at rate $\alpha$ if an infected individual meets a noninfected one, and from which infected individuals recover at rate $\mu$. Let $p$ be the proportion of infected individuals in a population; then $1 - p$ is the proportion of noninfected ones. Since infections require contact between infected and noninfected individuals, the increase of the proportion of infected individuals is proportional to both $p$ and $1 - p$; the constant of proportionality is $\alpha$. The loss of infected individuals is only proportional to $p$ with constant of proportionality $\mu$. Altogether, $p$ changes at rate

$$\dot{p} = \alpha p (1 - p) - \mu p$$

Subtask 1.3.1  Phase line diagrams, 1 point

Draw the phase line diagrams for $\alpha < \mu$ and $\alpha > \mu$. What follows for the qualitative behavior (equilibria, stability)? Sketch selected solutions.

Subtask 1.3.2  Discussion state of health, 1 point

Discuss what the two cases mean for the state of 'health' of the population and the spreading of the disease?

Exercise 1.4  Logistic ODE: Check solution of IVP, 1 point

Consider the logistic differential equation, this time in the form

$$\dot{x} = \lambda x \frac{K - x}{K}$$

Verify that the function

$$x(t) = \frac{K x_0}{x_0 + (K - x_0)e^{-\lambda t}}$$

is the solution of this differential equation with initial value $x_0$.

[Hint: Differentiate and have a sharp look at the resulting expression. Don't expand in any case!]

Exercise 1.5  Solution $x(t) = 2e^t - 1$ given: find IVP, 1 point

Find the initial value problem that is solved by $x(t) = 2e^t - 1$