Reasoning and Decision-Making under Uncertainty

3. Termin: Uncertainty, Degrees of Belief and Probabilities

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AG Sociable Agents

Intelligent agent

- Reasoning, inference
- Decision-making, action selection
Sources of uncertainty in reasoning

Epistemic limits
- knowledge about realistic domain is always approximative and simplified
- noisy sensors, partially observable environment, conflicting information

Representational limits
- notational adequacy of representation language
- frame problem (McCarthy & Hayes, 1969), qualification problem (McCarthy 1980), ramification problem

Inferential limits (e.g. of first-order logics)
- too many possible antecedents or consequents (incompleteness)
- no truth-preserving inferences (incorrectness)
- growth of uncertainty from (untested) antecedents to conclusions, especially when chaining inferences

Reasoning with uncertainties

Example: SARS diagnosis

Idea: Instead of enumerating all antecedents and conclusions, summarize them by numbers (e.g. probabilities)
Reasoning with uncertainties

Classical knowledge-based (or model-based) reasoning

Probabilistic reasoning

Vagueness vs. Uncertainty

Probabilities ~ uncertainty, if a proposition is true or not (→ degree of belief)
Not gradually true (vague) propositions (→ fuzzy-metrics)

Sources of uncertainty in decision-making

Stability and robustness limits
- environment dynamic
- environment non-deterministic (bounded or unbounden indeterminacy)

Complexity limits
- full deliberation too costly
- need for limited horizon of deliberation
- combinatorical explosion when accounting for contingencies and indeterminism

Decision-making with uncertainties

Let action $A_t = \text{leave for airport } t \text{ minutes before flight}$

**Question**: Will $A_t$ get me there on time?

What are the problems for a purely logical agent?

A purely logical approach either

- risks falsehood: “$A_{25}$ will get me there on time” = true
- leads to conclusions too weak and unreliable for decision-making

**Example**:

- $A_{90}$ will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact and .....  
  - plan success not inferrable (qualification problem)

Logical agent unable to act rationally under uncertainty!

**Idea**: rational decision depends on both relative importance of goals and likelihood that they will be achieved to the necessary degree
Decision-making with uncertainties

Idea in a nutshell

Use probabilistic assertions (not propositions) to summarize effects of
- laziness: failure to enumerate exceptions, qualifications, etc.
- ignorance: lack of relevant facts, initial conditions, etc.

Subjective probability relates facts to the own state of knowledge
- degree of belief, e.g., \( \Pr(A_{25} \mid \text{no reported accidents}) = 0.06 \)
- *not* a degree of truth, i.e. no assertions about the world, only about belief

Probabilities of assertions change when new evidence arrives
- posterior or conditional probabilities:
  \( \Pr(A_{25} \mid \text{no reported accidents, 5 a.m.}) = 0.15 \)

Suppose the agent believes the following:
- \( \Pr(A_{25} \text{ gets me there on time } \mid \ldots) = 0.04 \)
- \( \Pr(A_{90} \text{ gets me there on time } \mid \ldots) = 0.70 \)
- \( \Pr(A_{120} \text{ gets me there on time } \mid \ldots) = 0.95 \)
- \( \Pr(A_{1440} \text{ gets me there on time } \mid \ldots) = 0.999 \)

Which action to choose depends on preferences for possible outcomes (risks, costs, rewards, etc.), represented using utility theory
- decision theory = probability theory + utility theory

**Principle of maximum expected utility (MEU)**
An agent is rational iff it chooses the action that yields the highest expected utility, averaged over all possible outcomes of the action
Decision-making with uncertainties

**Idea in a nutshell**

```java
function DT-AGENT(percept) returns eine Aktion
    static: belief_state, probabilistischer Glauben über den aktuellen Zustand der Welt
        action, die Aktion des Agenten

    aktualisiere belief_state basierend auf action und percept
    berechne Ergebniswahrscheinlichkeiten für Aktionen
    abhängig von Aktionsbeschreibungen und aktuellem belief_state
    wähle action mit dem höchsten erwarteten Nutzen
    für gegebene Wahrscheinlichkeiten der Ergebnisse und Nutzeninformation
    return action
```

„Decision-theoretic Agent“

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Decision-making with uncertainties

**Classical knowledge-based (or model-based) decision-making**

- **Knowledge base (logics) + inferences**
- **Action selection (planning) based on true/false preconditions/effects**
- **Action(s)**
- **Goals**

**Probabilistic decision-making**

- **Probab. causal model (Bayesian network)**
- **Action selection based on expected utilities under current degrees of belief**
- **Action(s)**
- **Goals, Utilities**
Propositional logics

**World** = state of affairs in which each propositional variable is known

- variable assignment with values

**Models** = worlds that satisfy a sentence

- every sentence represents a set of worlds = (atomic) event

<table>
<thead>
<tr>
<th>World</th>
<th>Earthquake</th>
<th>Burglary</th>
<th>Alarm</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>true</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>w2</td>
<td>true</td>
<td>true</td>
<td>false</td>
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<tr>
<td>w3</td>
<td>true</td>
<td>false</td>
<td>true</td>
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<tr>
<td>w4</td>
<td>true</td>
<td>false</td>
<td>true</td>
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<td>w5</td>
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<td>true</td>
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<td>w6</td>
<td>false</td>
<td>true</td>
<td>false</td>
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<tr>
<td>w7</td>
<td>false</td>
<td>false</td>
<td>true</td>
</tr>
<tr>
<td>w8</td>
<td>false</td>
<td>false</td>
<td>false</td>
</tr>
</tbody>
</table>

\[ \text{Mods}(\alpha) = \{ \omega : \omega \models \alpha \} \]

\[ \text{Mods}(\alpha \land \beta) = \text{Mods}(\alpha) \cap \text{Mods}(\beta) \]

\[ \text{Mods}(\alpha \lor \beta) = \text{Mods}(\alpha) \cup \text{Mods}(\beta) \]

\[ \text{Mods}(\neg \alpha) = \overline{\text{Mods}(\alpha)} \]

Important properties of sentences

- consistent / satisfiable \[ \text{Mods}(\alpha) \neq \{\} \]
- valid \[ \text{Mods}(\alpha) \neq \Omega \quad \models \alpha \]

Important relationships of sentences

- equivalent \[ \text{Mods}(\alpha) = \text{Mods}(\beta) \]
- mutually exclusive \[ \text{Mods}(\alpha) \cap \text{Mods}(\beta) = \{\} \]
- exhaustive \[ \text{Mods}(\alpha) \cup \text{Mods}(\beta) = \Omega \]
- implies \[ \alpha \models \beta \] \[ \text{Mods}(\alpha) \subseteq \text{Mods}(\beta) \]
Monotonicity of logical reasoning

\[ \alpha : (\text{Earthquake} \lor \text{Burglary}) \Rightarrow \text{Alarm} \]
\[ \text{Mods}(\alpha) = \{\omega_1, \omega_3, \omega_5, \omega_7, \omega_8\} \]

\[ + \]
\[ \beta : \text{Earthquake} \Rightarrow \text{Burglary} \]
\[ \text{Mods}(\alpha \land \beta) = \text{Mods}(\alpha) \cap \text{Mods}(\beta) = \{\omega_1, \omega_5, \omega_7, \omega_8\} \]

Monotonicity
Learning new information can only rule out worlds:

- if \( a \) implies \( c \), then \((a \text{ and } b)\) will imply \( c \) as well

Especially problematic in light of qualification problem! (why?)

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Modeling degrees of belief as probabilities

Degree of belief or probability of a world
- in fuzzy logic, interpreted as possibility/vagueness
  (not the view adopted here)

Degree of belief or probability of a sentence

State of belief or joint probability distribution

<table>
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<tr>
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<th>( Pr(\cdot) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>true</td>
<td>true</td>
<td>true</td>
<td>.0190</td>
</tr>
<tr>
<td>w2</td>
<td>true</td>
<td>true</td>
<td>false</td>
<td>.0010</td>
</tr>
<tr>
<td>w3</td>
<td>true</td>
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<td>true</td>
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<tr>
<td>w8</td>
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<td>false</td>
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Properties of beliefs

Properties of (degrees of) beliefs
- bound
- baseline for inconsistent sentences
- baseline for valid sentences

Junctions of beliefs
- disjunction
- conjunction

\[ 0 \leq Pr(\alpha) \leq 1 \quad \forall \alpha \]
\[ Pr(\alpha) = 0 \quad \forall \alpha \text{ inconsistent} \]
\[ Pr(\alpha) = 1 \quad \forall \alpha \text{ valid} \]

\[ Pr(\alpha \lor \beta) = Pr(\alpha) + Pr(\beta) - Pr(\alpha \land \beta) \]
\[ Pr(\alpha \land \beta) = 0 \text{ if } \alpha, \beta \text{ mutually exclusive} \]

\[ Pr(\text{Earthquake} \land \text{Burglary}) = Pr(\omega_1) + Pr(\omega_2) = .02 \]
\[ Pr(\text{Earthquake} \lor \text{Burglary}) = .1 + .2 - .02 = .28 \]

Uncertainty and entropy

Entropy = quantifies uncertainty about a certain variable

\[ ENT(X) := - \sum_x Pr(x) \log_2 Pr(x) \]
\[ (0 \log 0 := 0) \]

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<tr>
<td>true</td>
<td>.1</td>
<td>.2</td>
<td>.2442</td>
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<tr>
<td>false</td>
<td>.9</td>
<td>.8</td>
<td>.7558</td>
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<tr>
<td>ENT(.)</td>
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<td>.722</td>
<td>.802</td>
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Updating beliefs

Evidence = a piece of information known to hold \( \beta \)

\( \rightarrow \) requires to update state of belief with certain properties

- accommodate evidence \( \Pr(\beta|\beta) = 1 \)
  \( \Pr(\omega|\beta) = 0 \) for all \( \omega \models \neg \beta \)

- normalized \( \sum_{\omega \models \beta} \Pr(\omega|\beta) = 1 \)

- retain impossible worlds \( \Pr(\omega) = 0 \rightarrow \Pr(\omega|\beta) = 0 \)

- retain relative beliefs in possible worlds
  \[
  \frac{\Pr(\omega)}{\Pr(\omega')} = \frac{\Pr(\omega|\beta)}{\Pr(\omega'|\beta)}
  \]
  \( \forall \omega, \omega' \models \beta, \Pr(\omega) > 0, \Pr(\omega') > 0 \)

Updating beliefs

\( \rightarrow \) update old state of beliefs through conditioning on evidence \( \beta \)

\[
\Pr(\omega|\beta) := \begin{cases} 
0 & \omega \models \neg \beta \\
\frac{\Pr(\omega)}{\Pr(\beta)} & \omega \models \beta 
\end{cases}
\]

new beliefs = old beliefs, normalized with old belief in new evidence

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<td>false</td>
<td>false</td>
<td>.7128</td>
</tr>
</tbody>
</table>

| Earthquake | Burglary | Alarm | \( \Pr(.|\text{Alarm}) \) |
|------------|---------|-------|--------------------------|
| true       | true    | true  | .0190/.2442              |
| true       | true    | false | 0                        |
| true       | false   | true  | .0560/.2442              |
| true       | false   | false | 0                        |
| false      | true    | true  | .1620/.2442              |
| false      | true    | false | 0                        |
| false      | false   | true  | .0072/.2442              |
| false      | false   | false | 0                        |

\[
\Pr(\text{Burglary}) = .2 \rightarrow \Pr(\text{Burglary}|\text{Alarm}) = .741
\]
Updating beliefs

More efficient: direct update of a sentence from new evidence through Bayesian conditioning

\[ Pr(\alpha|\beta) = \frac{Pr(\alpha \land \beta)}{Pr(\beta)} \]

follows from the following commitments

- worlds that contradict evidence have zero prob
- worlds that have zero prob continue to have zero prob
- worlds that are consistent with evidence and have positive prob will maintain their relative beliefs

Note: Bayesian conditioning is nothing else than application of the basic product rule

\[ Pr(\alpha \land \beta) = Pr(\alpha|\beta) \cdot Pr(\beta) \]

Example: State of belief from above

<table>
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<th>( Pr(\text{Alarm}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>true</td>
<td>.1</td>
<td>.2</td>
</tr>
</tbody>
</table>

Conditioning on first evidence: \( \text{Alarm}=\text{true} \)

| \( Pr(E|\text{Alarm}) \) | \( Pr(B|\text{Alarm}) \) | \( Pr(A|\text{Alarm}) \) |
|-------------------------------|------------------|-----------------|
| true | .307 | .741 | 1 |

Conditioning on second evidence: \( \text{Earthquake}=\text{true} \)

<table>
<thead>
<tr>
<th>( Pr(E\land E) )</th>
<th>( Pr(B\land E) )</th>
<th>( Pr(A\land E) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>true</td>
<td>1</td>
<td>.253</td>
</tr>
</tbody>
</table>

\( \rightarrow \) belief dynamics under incoming evidence is a consequence of the initial state of beliefs one has!
Summary

Problems of reasoning and decision-making under uncertainty

Vagueness vs. uncertainty

From propositional logics to probability theory
  ▶ degree of belief, state of belief = joint prob. distribution
  ▶ properties of beliefs
  ▶ belief updating (conditioning, Bayesian conditioning)